ASYMPTOTIC EXPANSIONS ASSOCIATED

WITH THE n-th POWER OF A DENSITY*

Richard A. Johnson

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University of Minnesota
Minneapolis, Minnesota

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I dedicate this thesis to my wife

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for her encouragement and assistance.

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Introduction and Summary.

This thesis deals primarily with asymptotic expansions which are associated with the n-th power of a density. More precisely, let X_n have a density which is proportional to $k(x)f^n(x)$ where $f(\cdot)$ has a unique mode m at which it is sufficiently smooth. It follows that $n^{\frac{1}{2}}(X_n-m)b$ converges in law to the standard normal where b is a suitable scaling constant. (see von Mises (1964), page 269, or Buehler (1965)). Focusing our attention on the sequence $n^{\frac{1}{2}}(X_n-m)b$, we find asymptotic expansions for the cumulative distribution function $F_n(\cdot)$ and its percentiles $\xi^{(n)}$ in terms of the limiting normal distribution.

A short review of asymptotic expansions can be found in Appendix A.1 together with further references on the subject. Wallace (1958) in a review article on the use of asymptotic expansions mentions three types of expansions. Although his remarks are primarily for the case of normalized sums of independent random variables, these expansions are also available in other cases - the first two, for example, are considered in this thesis. For certain sequences of distribution functions $\{F_n\}$ tending to the normal, Wallace first considers asymptotic expansions of $F_n(\cdot)$ in powers of $n^{-\frac{1}{2}}$. That is

$$F_n(\xi) \sim \Phi(\xi) + \sum_{j=1}^{\infty} n^{-j/2} \gamma_j(\xi) \qquad (n \to \infty)$$

where $\Phi(\cdot)$ is the standard normal distribution function. The second expansion is the "Cornish-Fisher" type of the form

$$\xi_\alpha^{(n)} \sim \xi_\alpha + \sum_{j=1}^{\infty} n^{-j/2} s_j(\xi_\alpha) \qquad (n \to \infty)$$

where $\xi_\alpha^{(n)}$ and ξ_α are α percentiles of F_n and of the normal distribution respectively. The third type of expansion can be obtained by inverting the last expression, and the resulting expansion is called a normalizing transformation. Work along this line has been done by Bol'shev (1959) and Wasow (1956). Bol'shev worked on sums of independent random variables, a case which in many instances fits into the framework of Wasow. Wasow established the existence of expansions of all three types assuming a particular form of asymptotic expansion for the derivative of the logarithm of the density. While the present thesis is concerned with the existence of similar expansions, our assumptions are different, and Appendix A.2 gives some examples where our expansions of the first two types are valid, but Wasow's regularity conditions are not satisfied.

In Chapter 1, we prove the existence of an expansion for $F_n(\cdot)$, the distribution function of $n^{\frac{1}{2}}(X_n-m)b$, in Theorem 1.5.3 and then find that the coefficients of the powers of $n^{-\frac{1}{2}}$ consist of polynomials times the normal density function. The polynomials have coefficients which are expressible in terms of $k(\cdot)$ and $\log f(.)$ together with their derivatives.

Chapter 2 establishes sufficient conditions for obtaining a valid asymptotic expansion for the percentiles from the expansion for the distribution function. Wasow (1956) has a general theorem of this type (using different assumptions as mentioned above), and Bol'shev discusses the case of normalized sums of independent random variables. In Section 2.2 an asymptotic expansion in powers of $n^{-\frac{1}{2}}$ is obtained for the percentiles of the variate $n^{\frac{1}{2}}(X_n-m)b$. The coefficients in the expansion are polynomials in the corresponding normal percentiles.

In Chapter 3, we concern ourselves with the posterior distribution associated with a sample of size n from the one parameter exponential family $C(\phi)\exp[\phi R(x)]$ when ϕ has a prior density $\rho(\phi)$ with respect to Lebesgue measure. The posterior distribution is centered at $\hat{\phi}$, the maximum likelihood estimate, and scaled by the square root of the Fisher information evaluated at $\hat{\phi}$ so as to be of the form $k(x)f^n(x)$ when $\sum_{i=1}^{n} R(x_i)/n = r$. If we fix r, as von Mises (1964) does for the Bernoulli situation, the expansions follow as above. However Theorem 3.1.1 does take into account the fact that r is only approximately fixed as n goes to infinity. Section 3.4 shows how the prior density enters the first few terms of the expansion of the standardized posterior distribution. The results of Chapter 3 are obtained by modifying the development of Chapter 1. Earlier work on the asymptotic normality of posterior distributions due to Bernstein (1934), Gnedenko (1962), and LeCam (1953,1958) (as well as von Mises) is discussed.

Notational Conventions.

A brief review of the definitions associated with asymptotic expansions is given in Appendix A.1. Further references are also given there. Our interest centers on the special case of asymptotic expansions which result as n, the power to which $f(\cdot)$ is raised, goes to infinity through the positive integers. An examination of the proofs of statements involving O-functions which depend on n will reveal that the O-relations hold for all sufficiently large values of n whether integer or not. However, if we restrict n to the positive integers, the O-relations can be asserted to hold for all n, thus avoiding the notational difficulty of keeping track of which values of n are large enough. This, together with the fact that we are considering a sequence of random variables, seems sufficient to justify the restriction of n to the positive integers. Therefore in this work we make the assumption that n takes on only positive integer values.

Throughout this work we will use the following notational conventions. $\Phi(\cdot)$ and $\varphi(\cdot)$ are the standard normal distribution function and density function defined by

$$\Phi(\xi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\xi} e^{-u^2/2}\, du$$

$$\varphi(\xi) = (2\pi)^{-\frac{1}{2}} e^{-\xi^2/2} \quad .$$

We also define ξ_α by

$$\Phi(\xi_\alpha) = \alpha$$

and $F_n(\cdot)$ denotes the distribution function of $n^{\frac{1}{2}}X_n$ where X_n has the density proportional to (1.1.7).

Chapter 1. An Asymptotic Expansion for a Distribution Function.

In this chapter we obtain an expansion for the distribution function. More particularly, let X_n have a density proportional to $k(x)f^n(x)$ where $f(\cdot)$ has a unique mode at m. Under suitable regularity conditions, $n^{\frac{1}{2}}(X_n-m)$ converges in law to a normal distribution. This can be seen from Laplace (1847), pages 400-403, von Mises (1964), Chapter VI, or Buehler (1965). It is shown below that the cumulative distribution function possesses an asymptotic expansion in terms of the limiting normal distribution. Section 1.1 shows that it is possible to scale so that the limit distribution is the standard normal. In Section 1.2 we show how to express the distribution function as the ratio of two integrals. Each of the two integrals is expanded and the quotient is formed. Theorem 1.5.3 gives the existence of the expansion of the quotient and in Section 1.6, the first few terms are calculated. These expressions require knowledge of the values of $k(\cdot)$ and $\log f(\cdot)$ as well as their derivatives evaluated at the mode m of $f(\cdot)$. Lastly, in Section 1.7, examples are given.

1.1. Formulation of the Problem.

Let $\{X_n\}_{n=1}^{\infty}$ be the sequence of random variables which is associated with the sequence of density functions of the form

$$(1.1.1) \qquad C_n k(x) f^n(x)$$

where $\{C_n\}_{n=1}^{\infty}$ is a sequence of positive constants, $k(\cdot)$ is a positive function, and $f(x)$ has a unique maximum at $x=m$ at which $f(\cdot)$ is sufficiently smooth. We can, without loss of generality, specialize to the case where $m=0$, $f(0)=1$, $f'(0)=0$, and $f''(0)=-1$. To see this, note that the centered random variables X_n-m have density functions

(1.1.2) $\qquad C_n k(x+m) f^n(x+m) \quad \text{each } n=1,2,\cdots$.

Now each of the density functions in (1.1.2) may be rewritten in the following form.

(1.1.3) $\qquad f^n(m) C_n k(x+m) \left[f(x+m)/f(m)\right]^n$

Consider then the sequence of random variables $\{Y_n\}_{n=1}^{\infty}$ defined by

(1.1.4) $\qquad Y_n = b(X_n - m) \quad \text{each } n=1,2,\cdots,$

where

$$b^2 = -f''(m)/f(m).$$

Now Y_n has density function

(1.1.5) $\qquad f^n(m) C_n b^{-1} k(x/b+m) \left[f(x/b+m)/f(m)\right]^n \quad \text{each } n=1,2,\cdots,$

which may be written as

(1.1.6) $\qquad C_n' k_1(x) f_1^n(x) \qquad \text{each } n=1,2,\cdots,$

where $k_1(x) = k(x/b+m)$, $f_1(x) = f(x/b+m)/f(m)$, and C_n' is the normalization constant. Notice that $f_1(x)$ has an absolute maximum at $x=0$ and that $f_1(0) = 1$, $f_1'(0) = 0$, and $f_1''(0) = -1$.

Since we can always transform the X_n according to (1.1.4), we assume at the outset that X_n has a density proportional to

(1.1.7) $\qquad k(x) f^n(x) \qquad \text{each } n=1,2,\cdots,$

where $f(x)$ has a maximum of unity which occurs at zero.

1.2. Method of Solution.

The procedure used to obtain the expansion is to consider $F_n(\xi)$ as being the ratio

(1.2.1) $$n^{\frac{1}{2}} \int_{-\infty}^{\xi n^{-\frac{1}{2}}} k(x) f^n(x)\,dx \Big/ n^{\frac{1}{2}} \int_{-\infty}^{\infty} k(x) f^n(x)\,dx,$$

to first find expansions for the numerator and denominator separately, and then to divide the expansions to obtain the desired result. The techniques used in Section 1.4 Lemmas 1.4.1 to 1.4.12 and Theorem 1.4.1, together with part of the proof of Theorem 1.5.1 in Section 1.5, are slight modifications of those given by de Bruijn (1961), Chapter 4. Some similar expansions have been obtained by Laplace. A discussion of Laplace's results can be found in Appendix A.4.

1.3. Basic Assumptions.

We shall assume at the outset that $C_1k(x)f(x)$ is the density function of a random variable. If, on the contrary, $N>1$ is the smallest integer such that $\int_{-\infty}^{\infty} k(x)f^N(x)dx < \infty$, then all the O-symbols in the asymptotic expansions must be modified to hold for $n>N$.

From the preceding discussion, we see that it is sufficient to consider the case where $f(0)=1$, $f'(0)=0$, and $f''(0)=-1$. The basic assumptions are stated for this case.

Assumption (i). $f(x)$ and $k(x)$ are analytic for $|x|\leq\delta_1$ where δ_1 is some positive constant.

Assumption (ii). $f(x)$ has an absolute maximum at $x=0$ and $f(x)\leq\rho_1<1$ for all real x with $|x|\geq\delta_1$.

We assume that $k(0)\neq 0$ and also that $f(x)\neq 0$ whenever $|x|\leq\delta_1$.

Suppose the conditions on $k(\cdot)$ and $f(\cdot)$ are satisfied for some δ_1 and ρ_1. We then find a smaller δ which works as well but for which an additional condition, relation (1.3.4), holds for $\log f(x)$.

Since $f(x)\neq 0$, $\log f(x)$ may be expanded in a Taylor series of the form

$$\log f(x) = -x^2/2 + \text{higher order terms} \quad .$$

Here we have used the fact that $f(0)=1$, $f'(0)=0$, and $f''(0)=-1$. Therefore there exists a δ_2 with $0<\delta_2<\delta_1$ such that

$$(1.3.1) \qquad \log f(x) \le -x^2/4 \quad \text{for} \quad -\delta_2 \le x \le \delta_2 .$$

Let $\delta_3 = \min(1, \delta_2)$. If for real x we let

$$\rho_2 = \sup_{\delta_3/2 \le |x| \le \delta_1} f(x)$$

we have $\rho_2 < 1$, since the supremum is attained, but unity is the absolute maximum of $f(x)$ on the real line.

Let

$$(1.3.2) \qquad \delta = \delta_3/2$$

and

$$(1.3.3) \qquad \rho = \max(\rho_2, \rho_1) \quad .$$

Note that $\delta < 1$. From (1.3.2), we see that not only are Assumptions (i) and (ii) satisfied, but in addition we have

$$(1.3.4) \qquad \log f(x) \le -x^2/4 \quad \text{for} \quad -\delta \le x \le \delta .$$

The first two chapters of this thesis deal with the situation where $k(\cdot)$ and $f(\cdot)$ satisfy the assumptions of this section. These assumptions are implicit in all the lemmas and theorems and will not be repeated each time.

1.4. Expansion of the Denominator.

Working with the δ and ρ given by expressions (1.3.2) and (1.3.3) in the Assumptions (i) and (ii), together with the condition given by (1.3.4), we obtain the desired expansion of the integral.

By the basic assumptions, $f(x)$ is positive for $|x| \le \delta_1$ where $\delta_1 \ge 2\delta$. It follows that for a fixed branch, $\log f(x)$ is also analytic. Therefore

$$(1.4.1) \qquad \log f(x) = \sum_{s=2}^{\infty} a_s x^s \qquad \text{for } |x| \le 2\delta$$

where

$$(1.4.2) \qquad s!a_s = \frac{d^s}{dx^s} \log f(x) \Big|_{x=0} \qquad \text{for } s=2,3,\cdots \ .$$

Here we have used the conditions $f(0)=1$ and $f'(0)=0$ to give $a_0=a_1=0$. We also have

$$(1.4.3) \qquad a_2 = -\tfrac{1}{2}$$

since $f''(0)= -1$.

Consider first the integrals over the intervals $(-\infty,-\delta)$ and (δ,∞).

Lemma 1.4.1. For every positive M

$$\left\{\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right\} k(x)f^n(x)dx = O(n^{-M}) \qquad (n>1) \ .$$

The O-function may depend on M.

Proof: Since by Assumption (ii), $f(x)\le\rho<1$ for $|x|\ge\delta$, we have for each fixed $M>0$

$$\left\{\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right\} k(x)f^n(x)dx \le \rho^{n-1} \int_{-\infty}^{\infty} k(x)f(x)dx$$
$$\le A_M n^{-M} \qquad \text{for } n>N_M.$$

The last inequality, being true for sufficiently large n, can be made to hold for all positive integer n by modifying the boundary constant.

By the discussion at the end of Appendix A.1, we see that the integrals in Lemma 1.4.1 contribute nothing to an asymptotic expansion in powers of $n^{-\frac{1}{2}}$.

We now proceed to further shorten the interval of interest by establishing the following lemma.

Lemma 1.4.2.

$$\left\{\int_{-\delta}^{-n^{-1/3}} + \int_{n^{-1/3}}^{\delta}\right\} k(x)f^n(x)dx = O(\exp\left[-n^{1/3}/4\right]) \quad (n > \delta^{-3}) .$$

Proof: By Assumption (i), $k(x)$ is analytic on $[-\delta,\delta]$ so $|k(x)| \le K < \infty$ for $|x| \le \delta$. Using (1.3.4), we find that

$$\left\{\int_{-\delta}^{-n^{-1/3}} + \int_{n^{-1/3}}^{\delta}\right\} k(x)f^n(x)dx \le K\left\{\int_{-\infty}^{-n^{-1/3}} + \int_{n^{-1/3}}^{\infty}\right\} \exp\left[-nx^2/4\right]dx$$

when $n > \delta^{-3}$. Also

$$\begin{aligned} n/4(x^2-n^{-2/3}) &= n/4(x+n^{-1/3})(x-n^{-1/3}) \\ &> n/4 \cdot n^{-1/3}(x-n^{-1/3}) \quad \text{for } x > n^{-1/3} \\ &> (x-n^{-1/3})/4 \qquad\qquad \text{for } x > n^{-1/3} \end{aligned}$$

or

$$\int_{n^{-1/3}}^{\infty} e^{-nx^2/4}\,dx \le e^{-n^{1/3}/4}\int_{n^{-1/3}}^{\infty} e^{-(x-n^{-1/3})/4}\,dx = 4e^{-n^{1/3}/4}$$

when $n > \delta^{-3}$. For the other integral, we make the substitution $y = -x$.

For later reference, we state the following lemma.

Lemma 1.4.3. For each fixed $N > 0$ and $b < 0$

$$\left\{\int_{-\infty}^{-n^{-1/3}} + \int_{n^{-1/3}}^{\infty}\right\} x^N e^{bnx^2} dx = O(\exp\left[bn^{1/3}/2\right]) \quad (n > 1) .$$

Proof: This result follows from the proof of Lemma 1.4.2 with the constant $-\frac{1}{4}$ in the exponent replaced by $b/2$. To see this, note that $|x|^N = O(\exp\left[-b|x|/2\right])$ all x so that $|x|^N \exp\left[bnx^2\right] = O(\exp\left[bnx^2/2\right])$ all x.

The problem has been reduced to a study of the interval $(-n^{-1/3}, n^{-1/3})$. Using (1.4.1) and (1.4.3) we write

$$k(x)f^n(x) = e^{-nx^2/2}\, k(x) \cdot \exp\left[n \sum_{s=3}^{\infty} a_s x^s\right] . \tag{1.4.4}$$

Regard the first factor on the right hand side of (1.4.4) as being the main factor and the second factor as being a function composed of the analytic functions

$$k(z) \; ,$$
$$\psi(z) = \sum_{s=3}^{\infty} a_s z^{s-3} \; , \tag{1.4.5}$$

and $\qquad w$

where $w = nx^3$ and $z=x$. That is, the second factor may be written as a particular evaluation of the function

$$P(w,z) = k(z)e^{w\psi(z)} \tag{1.4.6}$$

which is an analytic function of the two complex variables w and z in the region $\{|w| \leq 2, \; |z| \leq 2\delta\}$.

<u>Lemma 1.4.4.</u> If $P(w,z)$ is an analytic function of the two complex variables w and z in the region $\{|w| \leq 2, \; |z| \leq 2\delta\}$, then

$$P(w,z) = \sum_{\ell,m=0}^{\infty} c_{\ell m} w^{\ell} z^m \tag{1.4.7}$$

where the series converges absolutely in the region $\{|w| \leq 1, \; |z| \leq \delta\}$. Also the coefficients are given by

$$c_{\ell m} = \frac{1}{\ell!m!} \left[\frac{\partial^{\ell+m} P(w,z)}{\partial^{\ell} w \, \partial^m z}\right]_{w=0,z=0} \qquad \text{each } \ell,m=0,1,\cdots \; . \tag{1.4.8}$$

<u>Proof</u>: See Fuks (1963) pages 39-40 or Markushevich (1965) pages 101-105.

For any positive integer N we write

$$P_N(w,z) = \sum_{\substack{\ell,m \geq 0 \\ \ell+m \leq N}} c_{\ell m} w^{\ell} z^m \tag{1.4.9}$$

for the truncation of the double series given by (1.4.7).

<u>Lemma 1.4.5</u>. Let $P(w,z)$ be analytic in the region $\{|w|\le 2,\ |z|\le 2\delta\}$. Then, for each fixed positive integer N, there exist constants A_1 and A_2 such that

$$|P(w,z) - P_N(w,z)| \le A_1 |w|^{N+1} + A_2 |z|^{N+1}$$

in the region $\{|w|\le 1,\ |z|\le\delta\}$. The constants A_1 and A_2 may depend on N.

<u>Proof</u>: By Lemma 1.4.4, the double power series for $P(w,z)$ converges absolutely in the region $\{|w|\le 1,\ |z|\le\delta\}$. Therefore the individual terms are bounded. In particular

$$|c_{\ell m} 1^{\ell}\delta^{m}| \le M < \infty \quad \text{for some } M \text{ all } \ell, m \ge 0.$$

That is

$$|c_{\ell m}| \le M\delta^{-m} \quad \text{all } \ell, m \ge 0.$$

Let N be arbitrary but fixed. Now for $|w|\le 1/3$ and $|z|\le\delta/3$ we estimate

$$\left|\sum_{\ell+m>N} c_{\ell m} w^{\ell} z^{m}\right| \le M\left[\sum_{\ell+m>N} |z/\delta|^{m}\, |w/1|^{\ell}\right]$$

$$\le M\left[\sum_{k=N+1}^{\infty} (|w| + |z|/\delta)^{k}\right]$$

$$\le \frac{M}{1-2/3}(|w| + |z|/\delta)^{N+1}$$

$$\le 3M\left(2^{N+1}|w|^{N+1} + (2/\delta)^{N+1}|z|^{N+1}\right).$$

That is, there exist constants A_3 and A_4 such that

$$(1.4.10)\qquad |P(w,z)-P_N(w,z)| \le A_3|w|^{N+1} + A_4|z|^{N+1} \text{ for } |z|<\delta/3 \text{ and } |w|<1/3.$$

This result can be extended to the region $\{|z|\le\delta,\ |w|\le 1\}$. Since P and P_N are continuous over this region, $|P-P_N|$ is bounded by some constant M_1. In (1.4.10) we replace A_3 and A_4 by

$$A_1 = \max(A_3, M_1)/(1/3)^{N+1}$$

and

$$A_2 = \max(A_4, M_1)/(\delta/3)^{N+1}$$

respectively. Since N was arbitrary, the result follows.

Going back to equation (1.4.4), we have

$$k(x)f^n(x) = e^{-nx^2/2}\, P(nx^3, x) \tag{1.4.11}$$

$$= e^{-nx^2/2} \sum_{\ell,m=0}^{\infty} c_{\ell m}(nx^3)^{\ell} x^m \quad .$$

Here the double series converges absolutely for $|nx^3| \leq 1$ and $|x| \leq \delta$.

A few remarks are in order, dealing with the relationship between each of the $c_{\ell m}$ and the functions $k(\cdot)$ and $f(\cdot)$. By (1.4.8), each $c_{\ell m}$ depends on the partial derivatives of $P(w,z)$ evaluated at zero. Examining the form of $P(w,z)$ as given in (1.4.6), we see that the partial derivatives of $P(w,z)$ are in turn functions of $k(\cdot)$ and $\psi(\cdot)$. Now (1.4.5) defines $\psi(\cdot)$ as a power series so that its derivatives evaluated at zero are given by the relation

$$\psi^{(q)}(0) = q!\,a_{q+3} \qquad \text{each } q=0,1,2,\cdots \quad .$$

By (1.4.2), each a_s is a derivative of $\log f(\cdot)$ evaluated at zero which in turn is expressible as a function of the derivatives of $f(\cdot)$ evaluated at zero. Thus we could conceivably express each $c_{\ell m}$ as a function of the derivatives of $k(\cdot)$ and $f(\cdot)$ evaluated at zero. This, however, leads to very unwieldy expressions even for the first few terms. Table 1.6.1 does express the first few coefficients in terms of the derivatives of $k(\cdot)$ and $\log f(\cdot)$.

<u>Lemma 1.4.6.</u> There exists a constant A_3 such that

$$\int_{-n^{-1/3}}^{n^{-1/3}} e^{-nx^2/2} \left| P(nx^3, x) - P_N(nx^3, x) \right| dx \leq A_3 n^{-N/2-1} \quad \text{for } (n > \delta^{-3}).$$

each fixed positive integer N. The constant A_3 may depend on N.

Proof: Let N be an arbitrary but fixed integer. On the path of integration we have $|x| \leq n^{-1/3}$. Since we assume that $n > \delta^{-3}$ we have $|nx^3| \leq 1$ and $|x| \leq \delta$. Therefore by Lemma 1.4.5 there exist constants A_1 and A_2, which may depend on N, such that

$$(1.4.12) \qquad |P(nx^3,x)-P_N(nx^3,x)| \leq A_1 |nx^3|^{N+1} + A_2 |x|^{N+1} \text{ when } |x| \leq n^{-1/3} < \delta.$$

We then estimate

$$\int_{-n^{-1/3}}^{n^{1/3}} e^{-nx^2/2} |P-P_N| dx \leq A_1 \int_{-n^{-1/3}}^{n^{-1/3}} e^{-nx^2/2} |nx^3|^{N+1} dx + A_2 \int_{-n^{-1/3}}^{n^{-1/3}} e^{-nx^2/2} |x|^{N+1} dx$$

$$\leq 2A_1 n^{N+1} \int_0^\infty x^{3N+3} e^{-nx^2/2} dx + 2A_2 \int_0^\infty x^{N+1} e^{-nx^2/2} dx$$

$$\leq A_1 n^{-N/2-1} \Gamma\left(\frac{3N+3}{2}\right) 2^{3N/2+2} + A_2 n^{-N/2-1} \Gamma\left(\frac{N+1}{2}\right) 2^{-N/2-1}$$

$$\leq A_3 n^{-N/2-1}$$

for some A_3 with $n > \delta^{-3}$.

It will now be shown that the integral of the approximation $P_N \cdot \exp[-nx^2/2]$, over the outside of the interval $(-n^{-1/3}, n^{-1/3})$, is negligible.

Lemma 1.4.7. For each fixed positive integer N

$$\left\{ \int_{-\infty}^{-n^{-1/3}} + \int_{n^{-1/3}}^{\infty} \right\} |P_N| e^{-nx^2/2} dx = O(n^N \exp[-n^{1/3}/4]) \quad (n > 1).$$

Proof: Let N be arbitrary but fixed. Now

$$|P_N(nx^3,x)| \leq \sum_{\substack{\ell,m \geq 0 \\ \ell+m \leq N}} |c_{\ell m}| n^\ell |x|^{3\ell+m}$$

so that

$$\left\{\int_{-\infty}^{-n^{-1/3}} + \int_{n^{-1/3}}^{\infty}\right\}|P_N|e^{-nx^2/2}dx \le \sum_{\substack{\ell,m\ge 0\\ \ell+m\le N}} |c_{\ell m}|n^{\ell}\cdot 2 \int_{n^{-1/3}}^{\infty} x^{3\ell+m}e^{-nx^2/2}\,dx \quad .$$

The result follows from Lemma 1.4.3 with $b = -\frac{1}{2}$.

Combining the previous few lemmas we obtain the following lemma.

Lemma 1.4.8. For each fixed positive integer N

$$(1.4.13) \qquad \int_{-\infty}^{\infty} k(x)f^n(x)dx = \int_{-\infty}^{\infty} P_N(nx^3,x)e^{-nx^2/2}dx + O(n^{-N/2-1}) \; (n\ge 1).$$

The O-function is not necessarily uniform in N.

Proof: For a fixed but arbitrary integer N

$$(1.4.14) \quad \left| \int_{-\infty}^{\infty} k(x)f^n(x)dx - \int_{-\infty}^{\infty} P_N e^{-nx^2/2}\,dx \right|$$
$$\le \left|\left\{\int_{-\infty}^{-n^{-1/3}} + \int_{n^{-1/3}}^{\infty}\right\} k(x)f^n(x)dx\right|$$
$$+ \left\{\int_{-\infty}^{-n^{-1/3}} + \int_{n^{-1/3}}^{\infty}\right\} |P_N|\, e^{-nx^2/2}\,dx$$
$$+ \int_{-n^{-1/3}}^{n^{-1/3}} |P-P_N|e^{-nx^2/2}\,dx \quad .$$

By Lemmas 1.4.1, 1.4.3, 1.4.6, and 1.4.7 the right hand side of (1.1.14) is bounded by $An^{-N/2-1}$ for $n>q$ say.

Now the first integral on the left hand side of (1.1.14) is finite, by assumption, for $n=1,2,\cdots q$. For each fixed integer N, the second integral is a finite sum of finite integrals each $n=1,2,\cdots,q$. Therefore, by suitably modifying the bounding constant A, we can establish the desired result.

From Lemma 1.4.8 we obtain the asymptotic series.

Theorem 1.4.1.

$$\int_{-\infty}^{\infty} k(x)f^n(x)dx \sim \sum_{j=0}^{\infty} \beta_j n^{-\frac{1}{2}(j+1)} \qquad (n\ge 1)$$

where

$$(1.4.15) \qquad \beta_j = \begin{cases} 0 & \text{, if } j \text{ is odd} \\ 2^{\frac{1}{2}(j+1)} \sum_{r=0}^{j} c_{r,\,j-r} 2^r \Gamma(r+j/2+1/2), & \text{otherwise .} \end{cases}$$

The $c_{\ell m}$'s are given by (1.4.8).

Proof: By (1.4.13) we have for every fixed but arbitrary N

$$\int_{-\infty}^{\infty} k(x) f^n(x)dx = \sum_{\substack{\ell, m \geq 0 \\ \ell+m \leq N}} \int_{-\infty}^{\infty} P_N e^{-nx^2/2}\, dx + O(n^{-N/2-1}) \quad (n \geq 1)$$

$$= \sum_{\substack{\ell, m \geq 0 \\ \ell+m \leq N}} c_{\ell m}\, \varepsilon_{\ell+m}\, n^{-\frac{1}{2}(\ell+m+1)}\, 2^{\frac{1}{2}(3\ell+m+1)} \Gamma\left(\frac{3\ell+m+1}{2}\right)$$

$$+ O(n^{-N/2-1}) \quad (n \geq 1)$$

where

$$\varepsilon_{\ell+m} = \begin{cases} 1, & \text{if } \ell+m \text{ even} \\ 0, & \text{if } \ell+m \text{ odd .} \end{cases}$$

Combining coefficients having the same power of $n^{-\frac{1}{2}}$, say $n^{-(j+1)/2}$, we obtain the result.

This last theorem gives an asymptotic expansion for the normalization constant for the density function of $n^{\frac{1}{2}}X_n$.

1.5. The Expansion of the Distribution Function.

Writing

$$F_n(\xi) = P\left[n^{\frac{1}{2}} X_n \leq \xi\right]$$

$$= P\left[X_n \leq \xi n^{-\frac{1}{2}}\right]$$

or

$$(1.5.1) \qquad F_n(\xi) = \frac{\int_{-\infty}^{\xi n^{-\frac{1}{2}}} k(x) f^n(x)dx}{\int_{-\infty}^{\infty} k(x) f^n(x)dx}$$

we reduce the problem to finding an asymptotic expansion for

$$\int_{-\infty}^{\xi n^{-\frac{1}{2}}} k(x)f^n(x)dx$$

in powers of $n^{-\frac{1}{2}}$. A slight modification of the argument used above proves sufficient to provide such an expansion.

<u>Theorem 1.5.1</u>. For each fixed but arbitrary positive integer N and fixed ξ,

$$\int_{-\infty}^{\xi n^{-\frac{1}{2}}} k(x)f^n(x)dx = \int_{-\infty}^{\xi n^{-\frac{1}{2}}} e^{-nx^2/2}\, P_N(nx^3,x)dx + O(n^{-N/2-1}) \quad (n\geq 1)$$

where the O-symbol is uniform with respect to ξ for each N. However the bounding constant may depend on N together with $k(\cdot)$ and $f(\cdot)$.

<u>Proof</u>: For fixed but arbitrary integer N, consider the error in using the approximation when ξ is arbitrary but fixed.

$$(1.5.2) \qquad \left| \int_{-\infty}^{\xi n^{-\frac{1}{2}}} k(x)f^n(x)dx - \int_{-\infty}^{\xi n^{-\frac{1}{2}}} P_N e^{-nx^2/2}\, dx \right|$$

$$\leq \left| \int_{-\infty}^{-n^{-1/3}} k(x)f^n(x)dx - \int_{-\infty}^{-n^{-1/3}} e^{-nx^2/2}\, P_N dx \right|$$

$$+ \left| \int_{-n^{-1/3}}^{\xi n^{-1/2}} \left[k(x)f^n(x) - e^{-nx^2/2}\, P_N\right] dx \right|$$

Examining carefully the second integral on the right hand side of (1.5.2), we see that if $|\xi| n^{-1/2} \leq n^{-1/3}$

$$\left|\int_{-n^{-1/3}}^{\xi n^{-1/2}}\left[k(x)f^n(x)-P_N e^{-nx^2/2}\right]dx\right| \le \int_{-n^{-1/3}}^{\xi n^{-1/2}}\left|k(x)f^n(x)-P_N e^{-nx^2/2}\right|dx$$

$$\le \int_{-n^{-1/3}}^{n^{-1/3}}\left|k(x)f^n(x)-P_N e^{-nx^2/2}\right|dx$$

and if $\xi n^{-1/2} > n^{-1/3}$

$$\left|\int_{-n^{-1/3}}^{\xi n^{-1/2}}\left[k(x)f^n(x)-P_N e^{-nx^2/2}\right]dx\right| \le \int_{-n^{-1/3}}^{n^{-1/3}}\left|k(x)f^n(x)-P_N e^{-nx^2/2}\right|dx$$

$$+\left|\int_{n^{-1/3}}^{\xi n^{-1/2}}\left[k(x)f^n(x)-P_N e^{-nx^2/2}\right]dx\right|$$

$$\le \int_{-n^{-1/3}}^{n^{-1/3}}\left|k(x)f^n(x)-P_N e^{-nx^2/2}\right|dx$$

$$+\int_{n^{-1/3}}^{\infty}k(x)f^n(x)dx+\int_{n^{-1/3}}^{\infty}|P_N|e^{-nx^2/2}dx\,.$$

Clearly if $\xi n^{-1/2} < -n^{-1/3}$, we have

$$\left|\int_{-\infty}^{\xi n^{-1/2}}\left[k(x)f^n(x)-e^{-nx^2/2}\right]dx\right| \le \int_{-\infty}^{-n^{-1/3}}k(x)f^n(x)dx+\int_{-\infty}^{-n^{-1/3}}|P_N|e^{-nx^2/2}\,dx\,.$$

Combining these results, we have for any ξ

$$\left|\int_{-\infty}^{\xi n^{-1/2}}\left[k(x)f^n(x)-P_N e^{-nx^2/2}\right]dx\right| \le \int_{-\infty}^{-n^{-1/3}}k(x)f^n(x)dx+\int_{-\infty}^{-n^{-1/3}}|P_N|e^{-nx^2/2}dx$$

(1.5.3)
$$+\int_{-n^{-1/3}}^{n^{-1/3}}\left|k(x)f^n(x)-P_N e^{-nx^2/2}\right|dx$$

$$+\int_{n^{-1/3}}^{\infty}k(x)f^n(x)dx+\int_{n^{-1/3}}^{\infty}|P_N|e^{-nx^2/2}dx\,.$$

Assume that $n > \delta^{-3}$. By Lemmas 1.4.1, 1.4.2, 1.4.6, and 1.4.7, the right hand side of (1.5.3) is

$$O(n^{-M}) + O(\exp[-n^{1/3}/4]) + O(n^N \cdot \exp[-n^{1/3}/4]) + O(n^{-N/2-1}) \quad (n \to \infty) .$$

Here M is arbitrary, so we take $M=N$. We then have for all sufficiently large n and some constant A

$$(1.5.4) \qquad \left| \int_{-\infty}^{\xi n^{-\frac{1}{2}}} \left[k(x) f^n(x) - P_N e^{-nx^2/2} \right] dx \right| \le A n^{-N/2-1} \quad \text{all } \xi .$$

This can be extended to be true for all integer n by noting that the left hand side of (1.5.4) is bounded by

$$\int_{-\infty}^{\infty} k(x) f^n(x) dx + \int_{-\infty}^{\infty} |P_N| e^{-nx^2/2} dx$$

uniformly in ξ. Now both of these integrals are finite for integer $n \ge 1$. Therefore using the maximum of this sum over the integers where (1.5.4) does not hold, we can modify the constant A so that the result is valid for all integer values of n. Since N was arbitrary, the theorem is proved.

From Theorem 1.5.1, we obtain the complete asymptotic expansion for the numerator in (1.5.1).

<u>Theorem 1.5.2</u>. For each fixed ξ,

$$\int_{-\infty}^{\xi n^{-\frac{1}{2}}} k(x) f^n(x) dx \sim \sum_{j=0}^{\infty} \alpha_j(\xi) n^{-j/2-\frac{1}{2}} \qquad (n \to \infty)$$

where

$$(1.5.5) \qquad \alpha_j(\xi) = \sum_{r=0}^{j} c_{r,j-r} \int_{-\infty}^{\xi} y^{2r+j} e^{-y^2/2} dy \quad \text{for each } j = 0, 1, \cdots .$$

The $c_{\ell,m}$'s are given by (1.4.8).

<u>Proof</u>: The existence of the expansion follows from Theorem 1.5.1 and the α_j's are obtained by adding the coefficients of $n^{-\frac{1}{2}(j+1)}$. Here we have made the substitution $y = n^{\frac{1}{2}} \cdot x$ so as to obtain the form

of the coefficients given in (1.5.5).

By Theorems 1.4.1 and 1.5.2, we have the following expansions for fixed ξ and arbitrary N.

$$(1.5.6)\qquad \int_{-\infty}^{\infty} k(x)f^n(x)dx = \beta_0 n^{-\frac{1}{2}} + \cdots + \beta_N \cdot n^{-\frac{1}{2}(N+1)} + O(n^{-N/2-1}) \quad (n \geq 1)$$

and

$$(1.5.7)\qquad \int_{-\infty}^{\xi n^{-\frac{1}{2}}} k(x)f^n(x)dx = \alpha_0(\xi)n^{-\frac{1}{2}} + \cdots + \alpha_N(\xi)n^{-\frac{1}{2}(N+1)} + O(n^{-N/2-1}) \quad (n \geq 1).$$

Multiplying each of the expansions by $n^{\frac{1}{2}}$, we obtain for each fixed positive integer N

$$(1.5.8)\qquad n^{\frac{1}{2}} \int_{-\infty}^{\infty} k(x)f^n(x)dx = \beta_0 + \sum_{j=1}^{N} \beta_j n^{-j/2} + O(n^{-\frac{1}{2}(N+1)}) \quad (n \geq 1)$$

and

$$(1.5.9)\qquad n^{\frac{1}{2}} \int_{-\infty}^{\xi n^{-\frac{1}{2}}} k(x)f^n(x)dx = \alpha_0(\xi) + \sum_{j=1}^{N} \alpha_j(\xi)n^{-j/2} + O(n^{-\frac{1}{2}(N+1)}) (n \geq 1) .$$

We have $\beta_0 \neq 0$ since by (1.4.15) $\beta_0 = c_{00}2^{\frac{1}{2}}\Gamma(\frac{1}{2})$ and by (1.4.8) $c_{00} = k(0)$ which is different from zero by assumption. It follows from the argument given by Erdélyi (1956) Chapter 1, that the asymptotic expansion for a quotient such as

$$\frac{n^{\frac{1}{2}} \int_{-\infty}^{\xi n^{-\frac{1}{2}}} k(x)f^n(x)dx}{n^{\frac{1}{2}} \int_{-\infty}^{\infty} k(x)f^n(x)dx}$$

exists, and the coefficients of the powers of $n^{-\frac{1}{2}}$ may be obtained by formal substitution.

In fact, we make the assertion that for each fixed but arbitrary integer N

$$(1.5.10)\qquad F_n(\xi) = \gamma_0(\xi) + \cdots + \gamma_N(\xi)n^{-N/2} + O(n^{-\frac{1}{2}(N+1)}) \quad (n \to \infty)$$

where the O-function is uniform with respect to ξ . Although the result may be obtained by division of the expansions, it becomes lucid if we first find the expansion

$$(1.5.11)\qquad \frac{1}{n^{\frac{1}{2}}\int_{-\infty}^{\infty} k(x)f^n(x)dx} = \sum_{j=0}^{N} \beta'_j\, n^{-j/2} + O(n^{-\frac{1}{2}(N+1)}) \qquad (n \geq 1)$$

for each integer N (see Erdélyi (1956)) and then multiply (1.5.11) and (1.5.9) to obtain the expansion of the ratio. Recalling that the O-function in (1.5.9) is uniform in ξ, and noting that $\alpha_1(\xi),\cdots,\alpha_N(\xi)$ are bounded, we see that the O-function which arises as we collect terms is uniform in ξ. The following theorem has been proved.

<u>Theorem 1.5.3</u>. For any fixed real number ξ, there exists a sequence of constants $\{\gamma_j(\xi)\}_{j=0}^{\infty}$ such that for each integer N

$$(1.5.12)\qquad F_n(\xi) = \sum_{j=0}^{N} \gamma_j(\xi) n^{-j/2} + O(n^{-\frac{1}{2}(N+1)}) \qquad (n \geq 1)$$

where the O-function is uniform with respect to ξ.

Theorem 1.5.3 shows the existence of the complete asymptotic expansion for $F_n(\xi)$. We next concentrate on finding more information as to the form of each $\gamma_j(\cdot)$.

1.6. <u>The Form of the Coefficients $\gamma_j(\cdot)$</u>.

The sequence $\{\gamma_j(\xi)\}_{j=0}^{\infty}$ is obtained by formal division of two asymptotic series. That is, the sequence satisfies

$$(1.6.1)\qquad \alpha_j(\xi) = \sum_{r=0}^{j} \gamma_j(\xi)/\beta_{r-j} \quad \text{each } j=0,1,2,\cdots.$$

We begin by considering the sequences $\{\alpha_j(\xi)\}_{j=0}^{\infty}$ and $\{\beta_j\}_{j=0}^{\infty}$. By (1.4.15) and (1.5.5), the $\alpha_j(\xi)$ and β_j are given by

$$\beta_j = \begin{cases} 2^{\frac{1}{2}(j+1)} \sum_{r=0}^{j} c_{r,j-r}\, 2^r\, \Gamma(r+j/2+1/2), & k \text{ even} \\ 0 & , k \text{ odd} \end{cases}$$

and

$$a_j(\xi) = \sum_{r=0}^{j} c_{r,j-r} \int_{-\infty}^{\xi} y^{2r+j} e^{-y^2/2}\,dy \quad \text{each} \quad j=0,1,\cdots \quad .$$

The $c_{\ell,m}$ are given by (1.4.8).

Focusing our attention on the integrals

$$\int_{-\infty}^{\xi} y^s e^{-y^2/2}\,dy \tag{1.6.2}$$

which enter into the coefficients $\gamma_j(\xi)$ through the $a_j(\xi)$'s, we see that there is an arbitrary choice to make concerning their expression in terms of known functions. One choice would be to express the integrals as linear combinations of complete and incomplete gamma functions. However, it seems more natural to express the integrals in terms of the limiting distribution of $n^{\frac{1}{2}}X_n$. We adopt this latter procedure and find that upon repeated integration by parts, the integrals may be expressed in terms of the standard normal distribution, its derivatives, and polynomials in ξ.

We now proceed to evaluate $\gamma_0(\xi)$, $\gamma_1(\xi)$, $\gamma_2(\xi)$, and $\gamma_3(\xi)$ in terms of the $c_{\ell m}$ defined in (1.4.8).

Using (1.4.15) and evaluating the gamma functions, we find that

$$\beta_0 = 2^{\frac{1}{2}}\Gamma(\tfrac{1}{2})c_{00} = (2\pi)^{\frac{1}{2}} c_{00}$$

$$\beta_1 = 0$$

$$\begin{aligned}\beta_2 &= 2^{3/2}\left[c_{02}\,\Gamma(3/2) + 2c_{11}\,\Gamma(5/2) + 2^2c_{20}\,\Gamma(7/2)\right]\\ &= (2\pi)^{\frac{1}{2}}\left[c_{02} + 3c_{11} + 15c_{20}\right]\end{aligned}$$

and $\quad \beta_3 = 0 \quad .$

Using (1.5.5) and integrating by parts, we obtain the following expressions.

$$a_0(\xi) = (2\pi)^{\frac{1}{2}} c_{00}\Phi(\xi)$$

$$a_1(\xi) = -(2\pi)^{\frac{1}{2}}\varphi(\xi)\left[c_{10}(\xi^2+2) + c_{01}\right]$$

$$\begin{aligned}a_2(\xi) = &-(2\pi)^{\frac{1}{2}}\varphi(\xi)\left[c_{20}(\xi^5+5\xi^3+15\xi)+c_{11}(\xi^3+3\xi)+c_{02}\xi\right]\\ &+ (2\pi)^{\frac{1}{2}}\Phi(\xi)\left[15c_{20}+3c_{11}+c_{02}\right]\end{aligned}$$

and $\alpha_3(\xi) = -(2\pi)^{\frac{1}{2}}\varphi(\xi)\Big[c_{30}(\xi^8+8\xi^6+48\xi^4+192\xi^2+384)$

$$+ c_{21}(\xi^6+6\xi^4+24\xi^2+48)$$
$$+ c_{12}(\xi^4+4\xi^2+8)$$
$$+ c_{03}(\xi^2+2)\Big] \quad .$$

Therefore we find for the first four terms

$$\mathcal{V}_0(\xi) = \alpha_0(\xi)/\beta_0 = \Phi(\xi) \tag{1.6.3}$$

$$\mathcal{V}_1(\xi) = \alpha_1(\xi)/\beta_0 = -\varphi(\xi)c_{00}^{-1}\left[c_{10}(\xi^2+2)+c_{01}\right] \tag{1.6.4}$$

$$\mathcal{V}_2(\xi) = \alpha_2(\xi)/\beta_0 - \alpha_0(\xi)\beta_2/\beta_0^2$$

$$= -\varphi(\xi)c_{00}^{-1}\left[c_{20}\xi^5+(5c_{20}+c_{11})\xi^3+(15c_{20}+3c_{11}+c_{02})\xi\right] \tag{1.6.5}$$

and

$$\mathcal{V}_3(\xi) = \alpha_3(\xi)/\beta_0 - \alpha_1(\xi)\beta_2/\beta_0^2$$

$$= -\varphi(\xi)c_{00}^{-1}\Big\{c_{30}\xi^8+(8c_{30}+c_{21})\xi^6+(48c_{30}+6c_{21}+c_{12})\xi^4 \tag{1.6.6}$$

$$+\left[192c_{30}+24c_{21}+4c_{12}+c_{30}-c_{10}c_{00}^{-1}(c_{02}+3c_{11}+15c_{20})\right]\xi^2$$

$$+\left[384c_{30}+48c_{21}+8c_{12}+2c_{03}-c_{00}^{-1}(2c_{10}+c_{01})(c_{02}+3c_{11}+15c_{20})\right]\Big\}.$$

In the above calculations, we have used the integration by parts formulas

$$\int_{-\infty}^{\xi} y^{2r+j}e^{-y^2/2}dy = -e^{-\xi^2/2}\sum_{q=1}^{(2r+j+1)/2}\xi^{2r+j+1-2q}\prod_{s=1}^{q-1}(2r+j+1-2s), \text{ j odd}$$

and

$$\int_{-\infty}^{\xi} y^{2r+j}e^{-y^2/2}dy = -e^{-\xi^2/2}\sum_{q=1}^{(2r+j)/2}\xi^{2r+j+1-2q}\prod_{s=1}^{q-1}(2r+j+1-2s)$$

$$+ (2\pi)^{\frac{1}{2}}\Phi(\xi)\prod_{s=1}^{(2r+j)/2}(2r+j+1-2s), \text{ j even, } j>0.$$

The notation $\prod_{q=1}^{0}(2r+j+1-2q)$ is interpreted to mean unity.

We now investigate the form of the general term in the expansion.

Lemma 1.6.1. The coefficient $\mathcal{V}_j(\cdot)$, for $j>0$, does not contain a term involving $\Phi(\cdot)$.

Proof: We first write the coefficients $\alpha_j(\xi)$ and β_j in the form in which they are used.

$$(1.6.7) \quad \beta_j = \begin{cases} 0 & , \ j \text{ odd} \\ 2^{\frac{1}{2}(j+1)} \sum_{r=0}^{j} c_{r,j-r} 2^r \Gamma(r+j/2+1/2) & , \ j \text{ even} \end{cases}$$

$$(1.6.8) \quad \alpha_j(\xi) = \begin{cases} -(2\pi)^{\frac{1}{2}}\varphi(\xi) \sum_{r=0}^{j} c_{r,j-r}\left[\sum_{q=1}^{(2r+j+1)/2} \xi^{2r+j+1-2q} \cdot \prod_{s=1}^{q-1}(2r+j+1-2s)\right], & j \text{ odd} \\ -(2\pi)^{\frac{1}{2}}\varphi(\xi) \sum_{r=0}^{j} c_{r,j-r}\left[\sum_{q=1}^{(2r+j)/2} \xi^{2r+j+1-2q} \prod_{s=1}^{q-1}(2r+j+1-2s)\right] \\ \quad + \sum_{r=0}^{j} c_{r,j-r}(2\pi)^{\frac{1}{2}} \Phi(\xi) \prod_{s=1}^{(2r+j)/2} (2r+j+1-2s) & ; \ j \text{ even with } j>0. \end{cases}$$

By (1.6.3), we know that $\mathcal{V}_0(\xi) = \Phi(\xi)$. Also (1.6.4), (1.6.5), and (1.6.6) show that $\mathcal{V}_1(\xi)$, $\mathcal{V}_2(\xi)$, and $\mathcal{V}_3(\xi)$ do not involve $\Phi(\cdot)$.

The induction hypothesis is that $\mathcal{V}_s(\xi)$ for $s=1,2,\cdots,j-1$, does not have a term involving $\Phi(\xi)$. Since $\mathcal{V}_j(\xi)$ is obtained from the relation

$$\alpha_j(\xi) = \mathcal{V}_j(\xi)/\beta_0 + \sum_{s=1}^{j-1} \mathcal{V}_s(\xi)/\beta_{j-s} + \mathcal{V}_0(\xi)/\beta_j$$

or

$$(1.6.7) \qquad \mathcal{V}_j(\xi) = \beta_0^{-1}\left[\alpha_j(\xi) - \beta_j \mathcal{V}_0(\xi)\right] - \beta_0^{-1} \sum_{s=1}^{j-1} \mathcal{V}_s(\xi)/\beta_{j-s} ,$$

what we must prove is equivalent to showing that $\alpha_j(\xi) - \beta_j \mathcal{V}_0(\xi)$ does not have a term involving $\Phi(\xi)$. If j is odd, this result is clear from the expressions for $\alpha_j(\xi)$ and β_j. Therefore we need only investigate the case where j is even to complete the induction.

Substituting the expression for β_j and the relevant part of $\alpha_j(\xi)$ into $\alpha_j(\xi)-\beta_j\gamma_0(\xi)$, we reduce the proof to showing that

$$0 = \sum_{r=0}^{j} c_{r,j-r}(2\pi)^{\frac{1}{2}}\Phi(\xi)\cdot \prod_{q=1}^{\frac{2r+j}{2}}(2r+j+1-2q) - \sum_{r=0}^{j} c_{r,j-r}\, 2^{\frac{2r+j+1}{2}}\Gamma\left(\frac{2r+j+1}{2}\right)\cdot\Phi(\xi)$$

identically in the $c_{\ell m}$ for each even j.

This will be true if we have

$$(2\pi)^{\frac{1}{2}} \prod_{q=1}^{\frac{2r+j}{2}}(2r+j+1-2q) = 2^{\frac{2r+j+1}{2}}\Gamma\left(\frac{2r+j+1}{2}\right)$$

when j is even. That this equality is true is readily seen by using $\Gamma(z+1) = z\Gamma(z)$.

In fact we can say even more about the coefficients.

Lemma 1.6.2. The coefficient $\gamma_j(\xi)$ for $j>0$, consists of a polynomial times the standard normal density function.

Proof: Using (1.6.7) in an iterative manner, we see that γ_j is a linear combination of α_s where $s=0,1,\cdots,j$. But, by (1.6.8), the α's are of the form postulated except perhaps for terms containing $\Phi(\cdot)$. This last possibility is ruled out by Lemma 1.6.1.

As mentioned previously, the coefficients $c_{\ell m}$ which enter into the $\gamma_j(\xi)$ may be expressed as functions of the derivatives of $k(\cdot)$ and $\log f(\cdot)$. The coefficients $c_{\ell m}$ are defined by (1.4.8) and can be seen from (1.4.6) to be functions of $k(\cdot)$ and $\psi(\cdot)$ and their derivatives. By (1.4.5), the function $\psi(\cdot)$ is related to the derivatives of $\log f(\cdot)$. Table 1.6.1 gives the first few $c_{\ell m}$ in terms of $k(\cdot)$ and $\log f(\cdot)$. It was constructed by the method used in the example of the central order statistic given in Section 1.7. Using Table 1.6.1, it is possible to express γ_1, γ_2, and γ_3 in terms of the derivatives of $k(\cdot)$ and $\log f(\cdot)$.

The entry in row ℓ column m is $c_{\ell m}$. For example, $c_{02} = k''(0)/2$. In the table $k'' = k''(0)$ etc.

Table 1.6.1. Coefficients $c_{\ell m}$ in terms of derivatives of $k(\cdot)$ and $\log f(\cdot)$.

ℓ \ m	0	1	2	3
0	k	k'	$k''/2$	$k'''/6$
1	ka_3	$ka_4+k'a_3$	$ka_5+k'a_4+k''a_3/2$	
2	$ka_3^2/2$	$ka_3a_4 + k'a_3^2/2$		
3	$ka_3^3/6$			

1.7. Examples.

In this section, we shall consider two examples. First we consider the central order statistic and express the coefficients of powers of $n^{-\frac{1}{2}}$ in terms of the population distribution function and its derivatives. We then consider the t-distribution and receive a partial check by comparing our results with Fisher (1925) for this special case.

Let Z_n be the (λ_{n+1})-th order statistic from a sample of size $(\lambda+\mu)n+1$ where λ and μ denote positive integers. Let $G(\cdot)$ and $g(\cdot)$ be the population distribution function and density function respectively. Assume that

(1) there exists a z_o such that $g(z_o)>0$ and $G(z_o) = \frac{\lambda}{\lambda+\mu}$ where λ and μ are positive integers.

(2) $g(\cdot)$ is analytic in a neighborhood of z_o, say $|z_o| < \delta_1$ some $\delta_1 > 0$.

Now Z_n has a density which is proportional to

$$(1.7.1)\qquad g(z)G(z)^{\lambda n}\left[1-G(z)\right]^{\mu n}/G(z_o)^{\lambda n}\left[1-G(z_o)\right]^{\mu n} \quad .$$

If we consider the random variable $(Z_n-z_o)\cdot c$ where

$$(1.7.2)\qquad c^2 = g^2(z_o)\cdot(\lambda+\mu)^3/\lambda\cdot\mu \quad ,$$

we see that the basic assumptions are satisfied. Here we have made the identification

$$k(x) = g(x/c + z_o)$$

$$f(x) = G^{\lambda}(x/c+z_o)\left[1-G(x/c+z_o)\right]^{\mu}/G^{\lambda}(z_o)\left[1-G(z_o)\right]^{\mu}$$

Note that $k(x)f(x)$ is integrable so that all the asymptotic expansions will hold true for $n \geq 1$.

We now proceed to evaluate the first few terms in the expansion of $P(nx^3,x) = k(x)\exp\left[-nx^3\cdot\left(\sum_{s=3}^{\infty} a_s x^{s-3}\right)\right]$. Using the notation $P(w,z)$, we find by formal substitution

$$P(w,z) = \left[k+k'z+k''z^2/2+\cdots\right]\left[1+w\left(\sum_{s=3}^{\infty} a_s x^{s-3}\right)+w^2\left(\sum_{s=3}^{\infty} a_s x^{s-3}\right)^2/2+\cdots\right]$$

$$(1.7.3) \quad = k+k'z+ka_3w+k''z^2/2+(a_3k'+ka_4)zw+ka_3^2w^2/2+\cdots \quad .$$

Now

$$k = k(0) = g(z_o)$$
$$k' = k'(0) = g'(z_o)/b_2^{\frac{1}{2}} \cdot g(z_o)$$
$$k'' = k''(0) = g''(z_o)/b_2g^2(z_o)$$

where
$$b_2 = (\lambda+\mu)^2(\lambda^{-1}+\mu^{-1}) \quad .$$

In what follows, we write g' for $g'(z_o)$, etc. By straightforward differentiation of $\log f(\cdot)$, we find that

$$a_3 = (-3g'b_2 + 2g^2b_3)/6g^2b_2^{3/2}$$

and

$$a^4 = (-3g'^2b_2-4g''gb_2 + 12g^2g'b_3-6g^4b_4)/24\ g^4b_2^2$$

where

$$b_3 = (\lambda+\mu)^3/\lambda^2 - (\lambda+\mu)^3/\mu^2$$

and

$$b_4 = (\lambda+\mu)^4/\lambda^3 + (\lambda+\mu)^4/\mu^3 \quad .$$

These relations enable us to express the coefficients of the double power series in terms of the population distribution function and its derivatives. In particular upon substitution into (1.7.3), we obtain

$$(1.7.4) \quad \begin{cases} c_{00} = g \\ c_{10} = (-3g'b_2+2g^2b_3)/6gb_2^{3/2} \\ c_{01} = g'/b_2^{\frac{1}{2}}g \\ c_{20} = (9g'^2b_2^2-12g^2g'b_2b_3+4g^4b_3^2)/72g^3b_2^3 \\ c_{11} = (-15g'^2b_2-4g''gb_2+20g^2g'b_3-6g^4b_4)/24g^3b_2^2 \\ c_{02} = g''/2g^2b_2 \end{cases} \quad .$$

Using (1.6.3), (1.6.4), and (1.6.5), we obtain the first three terms of the asymptotic expansion.

$$\gamma_0(\xi) = \Phi(\xi)$$

$$(1.7.5) \qquad \gamma_1(\xi) = \varphi(\xi)\left[(g'/2g^2b_2^{\frac{1}{2}}-b_3/3b_2^{3/2})\xi^2-2b_3/3b_2^{3/2}\right]$$

$$\gamma_2(\xi) = \varphi(\xi)\left[\left(\frac{12g^2g'b_2b_3-4g^4b_3^2-9g'^2b_2^2}{72g^4b_2^3}\right)\xi^5\right.$$

$$+ (g''/6g^3b_2-5b_3^2/18b_2^3+b_4/4b_2^2)\xi^3$$

$$\left. + (3b_4/4b_2^2-5b_3^2/6b_2^3)\,\xi\right]$$

where $b_2=(\lambda+\mu)^2/\lambda+(\lambda+\mu)^2/\mu$, $b_3=(\lambda+\mu)^3/\lambda^2-(\lambda+\mu)^3/\mu^2$, and $b_4 = (\lambda+\mu)^4/\lambda^3 + (\lambda+\mu)^4/\mu^3$. It is understood that g and all its derivatives are all evaluated at z_o.

Summarizing, let z_o be determined by $G(z_o) = \lambda/(\lambda+\mu)$ where λ and μ are positive integers. Then $F_n(\cdot)$ is the distribution function of $n^{\frac{1}{2}}(X_{[n+1]} - z_o)g(z_o)b_2^{\frac{1}{2}}$ and

$$F_n(\xi) = \Phi(\xi) + \gamma_1(\xi)n^{-\frac{1}{2}} + \gamma_2(\xi)n^{-1} + 0(n^{-3/2}) \qquad (n\to\infty)$$

where $\gamma_1(\xi)$ and $\gamma_2(\xi)$ are given in (1.7.5).

If we consider the special case of the median where $\lambda=\mu=1$, the expansion simplifies to

$$(1.7.6) \qquad F_n(\xi) = \Phi(\xi) + \left[\varphi(\xi)g'/2^{5/2}g^2\right] n^{-\frac{1}{2}}$$

$$+ \varphi(\xi)\left[(-g'^2/64g^4)\xi^5 + (g''/48g^3-1/8)\xi^3\right.$$

$$\left.-(3/8)\xi\right]\, n^{-1} + 0(n^{-3/2}) \qquad (n\to\infty)$$

where z_o is determined by $G(z_o) = \frac{1}{2}$.

As a second example, let $k(x) = (1+x^2)^{-\frac{1}{2}}$ and $f(x) = (1+x^2)^{-\frac{1}{2}}$. In this case, the basic assumptions are satisfied and $n^{\frac{1}{2}}X_n$ has the t-distribution with n degrees of freedom.

We write

$$\log f(x) = -\tfrac{1}{2}\log(1+x^2) \tag{1.7.7}$$
$$= -\tfrac{1}{2}\left[x^2 - x^4/2 + \cdots\right]$$

from which we see that $a_3=0$, $a_4=1/4$, and $a_5=0$. The derivatives of k are found by straightforward calculation. Using Table 1.6.1, we obtain expressions for the $c_{\ell m}$ necessary to determine $\gamma_0(\xi)$, $\gamma_1(\xi)$, $\gamma_2(\xi)$, and $\gamma_3(\xi)$ from (1.6.3) through (1.6.6).

Thus the expansion for $F_n(\cdot)$ becomes

$$F_n(\xi) = \Phi(\xi) - \left[\varphi(\xi)(\xi^3+\xi)/4\right] n^{-1} + 0(n^{-3/2}) \quad (n \to \infty). \tag{1.7.8}$$

Now $F_n(\xi)$ is a t-distribution, and this expansion agrees with that given by Fisher (1925). Peiser (1949) has shown that the error is actually of the correct order.

Chapter 2. An Asymptotic Expansion of the Percentile.

In Section 2.1 we prove a general result giving sufficient conditions for the existence of an asymptotic expansion of the percentiles when it is known that the distribution function has an expansion satisfying certain regularity conditions. Similar theorems are given by Bol′shev (1959) and (1963), Dorogovcev (1962), Peiser (1949), and Wasow (1956).

Throughout the remainder of the chapter, we assume that X_n has a density proportional to $k(x)f^n(x)$ where $k(\cdot)$ and $f(\cdot)$ satisfy the basic assumptions given in Section 1.3. These conditions also are part of the theorems and lemmas from Sections 2.2 through 2.4, and they will not be written explicitly each time.

Focusing our attention on $n^{\frac{1}{2}}X_n$, whose distribution has an asymptotic expansion in terms of the standard normal, we ask whether the α^{th} percentile of the distribution of $n^{\frac{1}{2}}X_n$ has an asymptotic expansion in powers of $n^{-\frac{1}{2}}$ where the coefficients depend on the upper α^{th} percentile of the standard normal distribution. Let $\xi_\alpha^{(n)}$ be the upper α^{th} percentile of the distribution of $n^{\frac{1}{2}}X_n$. Then in particular, we want to obtain a sequence $\{\xi_j\}_{j=0}^{\infty}$ such that for every positive integer N, we have

$$\xi_\alpha^{(n)} = \xi_0 + \xi_1 n^{-\frac{1}{2}} + \cdots + \xi_N n^{-N/2} + 0(n^{-\frac{1}{2}(N+1)}) \quad (n \to \infty)$$

where $\xi_0, \ldots, \xi_N$ depend on α. The existence of such an expansion is proved in Section 2.2. In the following section the first three terms are calculated and in the last section, examples are given.

2.1. Existence of an Asymptotic Expansion for a Percentile.

In this section, we no longer restrict our attention to random variables with density functions proportional to the n-th power of some other density function. In fact, we consider any sequence of cumulative distribution functions possessing an asymptotic expansion having certain properties which are stated in the assumptions of Theorem 2.1.1 below. For instance, the normalized sums of independent, identically distributed random variables with finite third moments have such an expansion using the Hermite-Chebyshev polynomials.

Theorem 2.1.1. If

(1) $F_\infty(\cdot)$, $G_1(\cdot)$, $G_2(\cdot),\cdots$, are distribution functions of random variables

(2) for each integer N

$$G_n(\xi) = F_\infty(\xi) + \sum_{j=1}^{N} n^{-j/2}\, \psi_j(\xi) + O(n^{-(N+1)/2}) \quad (n \to \infty) \tag{2.1.1}$$

where the O-function is uniform in ξ

(3) for each fixed α $0<\alpha<1$, ξ_0 and $\xi^{(n)}$ are determined by

$$G_n(\xi^{(n)}) = \alpha \quad \text{and} \quad F_\infty(\xi_0) = \alpha \tag{2.1.2}$$

(4) $F'_\infty(\xi_0) \neq 0$

(5) $F_\infty(\cdot)$, $\psi_1(\cdot)$, $\psi_2(\cdot),\cdots$ are analytic in a neighborhood of ξ_0,

then there exist $\bar{\xi}_j(\alpha)$, $j=1,2,\cdots$, such that

$$\xi^{(n)} = \xi_0 + \sum_{j=1}^{N} \bar{\xi}_j(\alpha) n^{-j/2} + O(n^{-(N+1)/2}) \quad (n \to \infty) \tag{2.1.3}$$

for each positive integer N. That is, $\xi^{(n)}$ has a complete asymptotic expansion.

Before we begin the proof of the theorem we establish a lemma.

Lemma 2.1.1. Under the assumptions of Theorem 2.1.1, there exist $\bar{\xi}_j(\alpha)$ $j=1,2,\cdots$, such that

$$G_n(\tilde{\xi}_n) = F_\infty(\xi_0) + O(n^{-(N+1)/2}) \qquad (n \to \infty)$$

where $\tilde{\xi}_n = \xi_0 + \sum_{j=1}^{N} \xi_j(\alpha) n^{-\frac{1}{2}}$ each $N=1,2,\cdots$; :

Proof: Let N be arbitrary but fixed. Now $\psi_j(\xi)$ $j=1,2,\cdots,N$ has an expansion about ξ_0 for real ξ given by

$$(2.1.4) \qquad \psi_j(\xi) = \psi_j(\xi_0) + \cdots + \frac{1}{N!}\psi_j^{(N)}(\xi_0)(\xi-\xi_0)^N + A_\xi(j,N)|\xi-\xi_0|^{N+1}$$

for $|\xi - \xi_0| \le \delta_j$ some δ_j. Since $\psi_j^{(N+1)}(\cdot)$ is bounded on this interval, the $A_\xi(j,N)$ are bounded uniformly in j and ξ when $|\xi - \xi_0| \le \min_{j \le N} \delta_j$.

Put $\tilde{\xi}_n = \xi_0 + \xi_1 n^{-\frac{1}{2}} + \cdots + \xi_N n^{-N/2}$ where the ξ_j are constants to be determined later. For sufficiently large n, we have $|\tilde{\xi}_n - \xi_0| \le \min_{j \le N} \delta_j$, so that writing

$$(2.1.5) \qquad \tilde{\xi}_n - \xi_0 = \xi_1 n^{-\frac{1}{2}} + \cdots + \xi_N n^{-N/2}$$

and substituting into (2.1.4), we find that the expression for $\psi_j(\tilde{\xi}_n)$ becomes

$$\psi_j(\tilde{\xi}_n) = \psi_j(\xi_0) + \cdots + \psi_j^{(N)}(\xi_0)(\xi_1 n^{-1/2} + \cdots + \xi_N n^{-N/2})^N/N! + O(n^{-\frac{1}{2}(N+1)})(n \to \infty)$$

(2.1.6)

$$= a_{j0}(\xi_0) + a_{j1}(\xi_0)n^{-\frac{1}{2}} + \cdots + a_{jN}(\xi_0)n^{-N/2} + O(n^{-(N+1)/2}) \quad (n \to \infty)$$

for each $j=1,2,\cdots,N$. This last expression is arrived at by collecting coefficients of the same powers of $n^{-\frac{1}{2}}$. In a similar way, we obtain the expansion

$$F_\infty(\tilde{\xi}_n) = F_\infty(\xi_0) + F_\infty'(\xi_0)(\xi_1 n^{-\frac{1}{2}} + \cdots + \xi_N n^{-N/2}) + \cdots$$
$$+ F_\infty^{(N)}(\xi_0)(\xi_1 n^{-\frac{1}{2}} + \cdots + \xi_N n^{-N/2})^N/N! + O(n^{-(N+1)/2}) \quad (n \to \infty)$$

which can also be arranged in powers of $n^{-\frac{1}{2}}$ to give

$$(2.1.7) \qquad F_\infty(\tilde{\xi}_n) = F_\infty(\xi_0) + a_{\infty 1}(\xi_0)n^{-\frac{1}{2}} + \cdots + a_{\infty N}(\xi_0)n^{-N/2} + O(n^{-\frac{1}{2}(N+1)}) \quad (n \to \infty).$$

Note that since $F'_\infty(\xi_0) \neq 0$ by hypothesis, ξ_j enters the coefficient $a_{\infty j}(\xi_0)$ as the factor

$$F'_\infty(\xi_0)\,\xi_j$$

and is not present in $a_{\infty i}(\xi_0)$ for $i < j$. Now combining the results (2.1.6) and (2.1.7), we form the expansion

$$(2.1.8) \qquad F_\infty(\tilde{\xi}_n) + \sum_{j=1}^{N} n^{-j/2}\psi_j(\tilde{\xi}_n) = b_0 + b_1 n^{-\frac{1}{2}} + \cdots + b_N n^{-N/2} + O(n^{-\frac{1}{2}(N+1)})\ (n \to \infty)$$

where b_i, $i=1,2,\cdots,N$, is linear in ξ_i with coefficient equal to $F'_\infty(\xi_0)$ since the contribution comes from (2.1.7). (The b's depend of course on α (or equivalently on ξ_0), but for simplicity of notation, α is suppressed.) Every time ξ_i enters $n^{-j/2}\psi_j(\tilde{\xi}_n)$, it enters as a power of $\xi_i n^{-i/2}$ so that b_i is a function only of $\xi_0, \xi_1, \cdots, \xi_i$ and not $\xi_{i+1}, \cdots, \xi_N$ for each fixed α. Therefore we can solve the following system of equations by recalling that ξ_0 is uniquely determined by $F_\infty(\xi_0) = \alpha$.

$$(2.1.9) \qquad \begin{aligned} &b_1(\xi_0, \xi_1) = 0 \\ &b_2(\xi_0, \xi_1, \xi_2) = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &b_N(\xi_0, \xi_1, \cdots, \xi_N) = 0 \end{aligned}$$

Let $(\bar{\xi}_1(\alpha), \cdots, \bar{\xi}_N(\alpha))$ denote the solution of (2.1.9). Form

$$(2.1.10) \qquad \tilde{\tilde{\xi}}_n = \xi_0 + \bar{\xi}_1(\alpha) n^{-\frac{1}{2}} + \cdots + \bar{\xi}_N(\alpha) n^{-N/2}$$

and expand as above with $\tilde{\xi}_n$ replaced by $\tilde{\tilde{\xi}}_n$. The coefficients $b_1(\xi_0, \bar{\xi}_1(\alpha)), \cdots, b_N(\xi_0, \bar{\xi}_1(\alpha), \cdots, \bar{\xi}_N(\alpha))$ of the expansion of $G_n(\tilde{\tilde{\xi}}_n)$ are all zero. Since N was arbitrary in the above argument, we have proved the lemma.

<u>Proof of Theorem 2.1.1</u>: Let N be arbitrary but fixed. By assumption $F'_\infty(\xi_0) \neq 0$ and $F'_\infty(\cdot)$ is continuous in some neighborhood of ξ_0. Therefore we can find a δ such that for all ξ with

$|\xi - \xi_0| \le \delta$, we have $|F'_\infty(\xi_0)| \ge m > 0$ some constant m.

Now $G_n(\xi^{(n)}) = \alpha$ and by Lemma 2.1.1, $G_n(\tilde{\xi}_n) = \alpha + O(n^{-(N+1)/2})$. Therefore

$$(2.1.11) \qquad G_n(\xi^{(n)}) - G_n(\tilde{\xi}_n) = O(n^{-(N+1)/2}) \quad (n \to \infty) \ .$$

If we substitute from (2.1.1) for each term on the left and subtract, we obtain

$$(2.1.12) \qquad F_\infty(\xi^{(n)}) + \sum_{j=1}^{N} n^{-j/2}\psi_j(\xi^{(n)}) - F_\infty(\tilde{\xi}_n) - \sum_{j=1}^{N} n^{-j/2}\psi_j(\tilde{\xi}_n)$$
$$= O(n^{-(N+1)/2}) \quad (n \to \infty) \ .$$

Using the mean value theorem, the left hand side is equal to

$$(2.1.13) \qquad \left[F'_\infty(\eta_n) + \sum_{j=1}^{N} n^{-j/2}\psi_j(\eta_n)\right](\xi^{(n)} - \tilde{\xi}_n)$$

where η_n lies between $\tilde{\xi}_n$ and $\xi^{(n)}$ each n. Here we have assumed that n is large enough so that both $\xi^{(n)}$ and $\tilde{\xi}_n$ are in the domain of analyticity of the approximation on the right hand side of (2.1.1).

Now n may be increased if necessary to make $|\tilde{\xi}_n - \xi_0| \le \delta$ and $|\xi^{(n)} - \xi_0| \le \delta$, since $\xi^{(n)}$ also converges to ξ_0. We then have $|F'_\infty(\eta_n)| \ge m$ and using the fact that $\psi_j(\cdot)$ is bounded for each $j=1,2,\cdots,N$ in a neighborhood of ξ_0, we deduce that the first factor in the expression (2.1.13) is greater than m/2 for sufficiently large n.

By (2.1.12) and (2.1.13), there exists an A such that

$$(2.1.14) \qquad An^{-(N+1)/2} \ge |\xi^{(n)} - \tilde{\xi}_n|\, m/2$$

for sufficiently large n. Since N was arbitrary, we have the desired result.

Now in the special case where $F'_\infty(\xi) > 0$ and the functions of interest are analytic in some neighborhood of ξ for each real ξ, the percentile expansion exists for all real ξ .

2.2. Application to a Particular Case.

We now return to the situation where $G_n(\cdot)$ equals $F_n(\cdot)$, the distribution function of $n^{\frac{1}{2}}X_n$ with X_n having a density of the form $k(x)f^n(x)$ satisfying the basic assumptions. Theorem 1.5.3 gives the asymptotic expansion of $F_n(\cdot)$. In this case, the limiting distribution is $\Phi(\cdot)$, which has a positive derivative for all real ξ. Also $\mathscr{V}_j(\cdot) = \mathcal{V}_j(\cdot)$ each $j=1,2,\cdots$ and by Lemma 1.6.2, each $\mathcal{V}_j(\cdot)$ consists of $\varphi(\xi)$ times a polynomial in ξ. That is, the assumptions of Theorem 2.1.1 are satisfied for each real ξ. We have then the following theorem.

Theorem 2.2.1. If for each fixed α with $0<\alpha<1$, $\xi_\alpha^{(n)}$ and ξ_α are determined by

$$1-\alpha = \Phi(\xi_\alpha)$$

and

$$1-\alpha = F_n(\xi_\alpha^{(n)}) \quad ,$$

then there exist $\bar{\xi}_1(\alpha)$, $\bar{\xi}_2(\alpha)$, $\cdots$, such that

$$\xi_\alpha^{(n)} = \xi_\alpha + \sum_{j=1}^{N} \bar{\xi}_j(\alpha) + O(n^{-(N+1)/2}) \quad (n \to \infty)$$

for each $N=1,2,\cdots$.

The $\bar{\xi}_j(\alpha)$ are obtained by the method used in the proof of Lemma 2.1.1. More particularly, they are obtained as a solution of the system of equations (2.1.9).

We note that Lemma 2.1.1 shows that

$$\tilde{\bar{\xi}}_N(\alpha) = \xi_\alpha + \sum_{j=1}^{N} \bar{\xi}_j(\alpha) n^{-j/2} \tag{2.2.1}$$

is a modified upper α^{th} percentile for $F_n(\cdot)$.

Theorem 2.2.2. Under the assumptions of the previous theorem,

$$F_n(\tilde{\bar{\xi}}_N(\alpha)) = 1-\alpha + O(n^{-\frac{1}{2}(N+1)}) \quad (n \to \infty)$$

for each $N=1,2,\cdots$.

We also have information as to the structure of $\bar{\xi}_j(\alpha)$, $j=1,2,\cdots$ given by the following theorem.

Theorem 2.2.3. Under the assumptions of Theorem 2.2.1, $\bar{\xi}_j(\alpha)$ is a polynomial in ξ_α each $j=1,2,\cdots$.

Proof: In the notation of equation (2.1.1), we have $F_\infty(\xi) = \Phi(\xi)$ and $\psi_j(\xi) = \varphi(\xi)p_j(\xi)$ each $j=1,2,\cdots$ where $p_j(\xi)$ is a polynomial in ξ . Therefore all the derivatives of $\Phi(\cdot)$ and $\psi_j(\cdot)$ each $j=1,2,\cdots$ consist of the standard normal density times a polynomial. From the general expansions which lead to (2.1.6) and (2.1.7) and then to (2.1.8), it can be seen that in our special case, each b_j consists of $\varphi(\xi_\alpha)$ times a linear combination of terms of the form $\xi_\alpha^{q_0}\cdots\xi_j^{q_j}$ where q_i is a non-negative integer. Recall that ξ_j enters b_j only as $\varphi(\xi_\alpha)\cdot\xi_j$. Therefore the corresponding equation in (2.1.9) expresses $\bar{\xi}_j(\alpha)$ as a linear combination of terms of the form $\xi_\alpha^{q_0}\bar{\xi}_1^{q_1}(\alpha)\cdots\bar{\xi}_{j-1}^{q_{j-1}}(\alpha)$ where q_i, $i=0,1,\cdots,j-1$, is a non-negative integer. A proof by induction establishes the desired result.

2.3. Calculation of the First Three Terms.

In the previous section, namely Theorem 2.2.1, an asymptotic expansion for $\xi_\alpha^{(n)}$ is shown to exist. In this section, we give explicit formulas for the first three terms of the expansion of $\xi_\alpha^{(n)}$.

Using the expansion method contained in the proof of Lemma 2.1.1, we obtain from (2.1.7) the expansion of $\Phi(\cdot)$. Let $\tilde{\xi}_n = \xi_0 + \xi_1 n^{-\frac{1}{2}} + \xi_2 n^{-1}$. Then

$$\Phi(\tilde{\xi}_n) = \Phi(\xi_0) + \varphi(\xi_0)(\xi_1 n^{-\frac{1}{2}} + \xi_2 n^{-1}) - \tfrac{1}{2}\xi_0\varphi(\xi_0)\cdot\xi_1^2 n^{-1} + o(n^{-3/2}) \quad (n\to\infty).$$

or

$$(2.3.1) \qquad \Phi(\tilde{\xi}_n) = \Phi(\xi_0) + \varphi(\xi_0)\left[\xi_1 n^{-\frac{1}{2}} + (\xi_2 - \xi_0\xi_1^2/2)n^{-1}\right] + o(n^{-3/2}) \quad (n\to\infty).$$

Next we find the expansions for $\gamma_1(\tilde{\xi}_n)$ and $\gamma_2(\tilde{\xi}_n)$. Now (1.6.4) and (1.6.5) are the expressions for $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ which are used to obtain expansions analogous to (2.1.6). After straightforward manipulations, we find that

$$(2.3.2) \quad \gamma_1(\tilde{\xi}_n) = \varphi(\xi_0)c_{00}^{-1}\left[-c_{10}\xi_0^2 - 2c_{10} - c_{01} + n^{-\frac{1}{2}}(c_{10}\xi_0^3\xi_1 + c_{01}\xi_0\xi_1)\right] + O(n^{-1}) \quad (n \to \infty)$$

and

$$(2.3.3) \quad \gamma_2(\tilde{\xi}_n) = \varphi(\xi_0)c_{00}^{-1}\left[-c_{20}\xi_0^5 - (5c_{20}+c_{11})\xi_0^3 - (15c_{20}+3c_{11}+c_{02})\xi_0\right] + O(n^{-\frac{1}{2}}) \quad (n \to \infty).$$

From (2.3.1), (2.3.2), and (2.3.3), we obtain the expansion corresponding to (2.1.8) with N=2.

$$(2.3.4) \quad F_n(\tilde{\xi}_n) = \Phi(\xi_0) + \varphi(\xi_0)\left[b_1 n^{-\frac{1}{2}} + b_2 n^{-1}\right] + O(n^{-3/2}) \quad (n \to \infty)$$

where

$$b_1 = \xi_1 - c_{00}^{-1}(c_{10}\xi_0^2 + 2c_{10} + c_{01})$$

and

$$b_2 = \xi_2 - \xi_1^2\xi_0/2 + c_{00}^{-1}(c_{10}\xi_0^3\xi_1 + c_{01}\xi_0\xi_1) - c_{00}^{-1}\left[c_{20}\xi_0^5 + (5c_{20}+c_{11})\xi_0^3 + (15c_{20}+3c_{11}+c_{02})\xi_0\right]$$

Consequently the desired quantities are

$$(2.3.5) \quad \bar{\xi}_0(\alpha) = \xi_\alpha \quad \text{where } 1-\alpha = \Phi(\xi_\alpha)$$

$$(2.3.6) \quad \bar{\xi}_1(\alpha) = c_{00}^{-1}(c_{10}\xi_\alpha^2 + 2c_{10} + c_{01})$$

and

$$(2.3.7) \quad \bar{\xi}_2(\alpha) = (c_{20}c_{00}^{-1} - c_{10}^2c_{00}^{-2}/2)\,\xi_\alpha^5 + (5c_{20}c_{00}^{-1} + c_{11}c_{00}^{-1} - c_{01}c_{10}c_{00}^{-2})\,\xi_\alpha^3 + (2c_{10}^2c_{00}^{-2} - c_{01}^2c_{00}^{-2}/2 + 15c_{20}c_{00}^{-1} + 3c_{11}c_{00}^{-1} + c_{02}c_{00}^{-1})\,\xi_\alpha .$$

A closer examination of the relationship between c_{10} and c_{20}, as given in Table 1.6.1, reveals that the coefficient of ξ_α^5 in (2.3.7) is zero. Therefore (2.3.7) reduces to

$$(2.3.8) \qquad \xi_2(\alpha) = (5c_{20}c_{00}^{-1}+c_{11}c_{00}^{-1}-c_{01}c_{10}c_{00}^{-2})\xi_\alpha^3$$
$$+ (2c_{10}^2c_{00}^{-2}-c_{01}^2c_{00}^{-2}/2 + 15c_{20}c_{00}^{-1}+3c_{11}c_{00}^{-1}+c_{02}c_{00}^{-1})\xi_\alpha .$$

2.4. Examples of the Percentile Expansion.

A particular example of the previous section occurs when we consider a central order statistic. Using the notation of Section 1.7 and the results (1.7.4), we find the first three terms of the expansion for the upper α^{th} percentile of the random variable

$$n^{\frac{1}{2}}\left[g^2(z_o)(\lambda+\mu)^3/\lambda\mu\right]^{\frac{1}{2}}(Z_n - z_o) .$$

Substituting the relations (1.7.4), which give $c_{\ell m}$ in terms of the derivatives of the population, we obtain from (2.3.5), (2.3.6), and (2.3.8) the desired result.

$$(2.4.1) \qquad \xi_0(\alpha) = \xi_\alpha \quad \text{where } 1-\alpha = \Phi(\xi_\alpha)$$

$$(2.4.2) \qquad \xi_1(\alpha) = \left[\frac{-3g'b_2+2g^2b_3}{3!g^2b_2^{3/2}}\right]\xi_\alpha^2 + \frac{2b_3}{3b_2^{3/2}}$$

$$(2.4.3) \qquad \xi_2(\alpha) = (-g''/6g^3b_2+5b_3^2/18b_2^3-b_4/4b_2^2+g'^2/2g^4b_2-g'b_3/3g^2b_2^2)\xi_\alpha^3$$
$$+ (-3b_4/4b_2^2+19b_3^2/18b_2^3-2g'b_3/3g^2b_2^2)\xi_\alpha$$

In the particular case of the median in a sample of size 2n+1, there is further simplification since $b_2=2^3$, $b_3=0$, and $b_4=2^5$. In this case

$$(2.4.4) \qquad \xi_\alpha^{(n)} = \xi_\alpha-n^{-\frac{1}{2}}\left[\frac{g'}{g^2 2^{5/2}}\right]\xi_\alpha^2 + n^{-1}\left[(-g''/48g^3-1/8+g'^2/16g^4)\xi_\alpha^3-(3/8)\xi_\alpha\right]$$
$$+ O(n^{-3/2}) \quad (n\to\infty) .$$

Again as a partial check on our calculations, we consider the t-distribution discussed in Section 1.7. Using the results of that section, we obtain upon direct substitution into (2.3.5), (2.3.6), and (2.3.8)

$$\xi_\alpha^{(n)} = \xi_\alpha + n^{-1}(\xi_\alpha^3 + \xi_\alpha)/4 + O(n^{-3/2}) \quad (n \to \infty) . \tag{2.4.5}$$

This checks with Peiser (1949).

Chapter 3. Posterior Distributions.

Let $\emptyset$ be the real valued parameter for the exponential family having densities of the form

$$p_{\emptyset}(x) = C(\emptyset)\exp\left[\emptyset\, R(x)\right] \tag{3.0.1}$$

with respect to a σ-finite measure μ over a Euclidean sample space. In order to avoid a trivial case, we assume that

$$R(x) \neq \text{const.} \quad (\text{a.s. } \mu). \tag{3.0.2}$$

Now assume that the parameter $\emptyset$ has a prior density $\rho(\emptyset)$. The posterior density of $\emptyset$ given $(X_1,\cdots,X_n) = (x_1,\cdots,x_n)$ is proportional to

$$\left[C(\emptyset)e^{\emptyset r}\right]^n \rho(\emptyset) \quad \text{where } r = \sum_{i=1}^{n} R(x_i)/n. \tag{3.0.3}$$

The expression (3.0.3) is proportional to a density function and hence defines a random variable $\emptyset$ whose density depends on r. But since $r = \sum_{i=1}^{n} R(x_i)/n$, the distribution of X given $\emptyset = \emptyset_o$ generates a sequence $(x_1,x_2,\cdots)$ and a sequence $(R(x_1), \frac{1}{2}\left[R(x_1)+R(x_2)\right],\cdots)$ and ultimately an infinite sequence of posterior densities of $\emptyset$. It is the asymptotic form of this sequence with which we shall be concerned.

Exploiting the relationship between the density proportional to (3.0.3) and the density considered in Chapter 1, we obtain an asymptotic expansion in much the same way as before. In this case, however, $r = \sum_{i=1}^{n} R(x_i)/n$ has a random aspect coming from the observed sequence $(x_1,x_2,\cdots)$. In order to show that the asymptotic expansions exist for almost all sequences $(x_1,x_2,\cdots)$, we show that conditions analogous to those in Chapter 1 hold uniformly in r in some neighborhood.

Neglecting for the moment the stochastic aspect of r, we see that if r is held fixed perhaps at the expected value of $R(x)$, the density proportional to (3.0.3) is exactly of the form considered in Chapter 1, and accordingly we obtain an asymptotic expansion. Closely related to this approach is the work by Bernstein and von Mises. Their results are for the Bernoulli situation, and both use the usual parameter p rather than the $\phi = \log(p/(1-p))$ which results if the density is cast into the form (3.0.3). von Mises actually holds r fixed as he passes to the limit. Their results, which give only the limiting normal term, are reproduced in Bernstein (1934), page 406 and von Mises (1964), Chapter VII, Section C. von Mises also gives the multinomial generalization. A more recent work following the same line of attack is given in Gnedenko (1962), Section 65. Gnedenko considers three cases corresponding to various combinations of mean and variance known and unknown when the likelihood is normal. He shows that the limiting distribution is normal. Gnedenko also gives expansions for the density functions and posterior mean and variance. In all these works, the stochastic aspect of r is ignored.

LeCam, in two basic papers (1953) and (1958), does take into account the stochastic nature of $(x_1, x_2, \cdots)$, and his Theorem 7 (1953) and Lemma 5 (1958) show that under very general conditions, the scaled posterior distribution converges to the normal distribution for almost all sequences $(x_1, x_2, \cdots)$ with respect to the infinite product measure generated by (3.0.1). His more general conditions include both a more general likelihood than ours and the case where the parameter is multidimensional. See LeCam (1953) page 278, for a discussion concerning the historical background on the problem of convergence of the scaled posterior distribution. Further connections with LeCam's work are discussed below at the end of Section 3.3.

The main theorem of this chapter is given in Section 3.1 and the following two sections give the details needed to modify the approach of Chapter 1 so that it works in the present situation. Section 3.4 gives the first two correction terms of the expansion together with examples.

3.1. A Limit Theorem for Posterior Distributions.

Let $X_1, X_2, \dots$ be independent, identically distributed random variables with density function $p_\phi(\cdot)$ given by (3.0.1). Lehmann (1959), Section 2.7, has shown that $p_\phi(\cdot)$ is a probability density function for all ϕ belonging to some interval I. Nature chooses a value ϕ_o for ϕ according to law $\rho(\phi)$, and we assume ϕ_o is interior to I. This will always be the case when I is open and in any case, will have prior probability one.

It is shown in Lemma 3.2.1 that $C(\phi)e^{\phi r}$ has a maximum at $\hat{\phi}(r)$ for fixed r when r lies in some neighborhood of $E_{\phi_o} R$. According to (3.0.3), this implies that the posterior distribution of ϕ given $\Sigma R(x_i)/n = r$ has a mode at $\hat{\phi}(r)$. We proceed to study the asymptotic form of the posterior distribution of ϕ after standardizing ϕ by centering at the mode and scaling according to the curvature at the mode. Thus we introduce

$$(3.1.1) \qquad \theta = \left[\phi - \hat{\phi}(r)\right] b(r)$$

where

$$(3.1.2) \qquad b^2(r) = \left[C'(\hat{\phi})^2 - C''(\hat{\phi})C(\hat{\phi})\right]/C^2(\hat{\phi}), \quad \hat{\phi} = \hat{\phi}(r).$$

Note that $\hat{\phi}(r)$ is the maximum likelihood estimate and $b^2(r)$ is the Fisher information evaluated at $\hat{\phi}(r)$. The choice of normalizing transformation (3.1.1) is discussed below in Section 3.3 after the proof of Theorem 3.1.1.

Denote the posterior distribution function of $n^{\frac{1}{2}}\theta$ by $F_n(\cdot,r)$. Recall that we observe a sequence $(x_1,x_2,\cdots)$ which generates a sequence $(r_1,r_2,\cdots)$ where $r_n = \sum_{i=1}^{n} R(x_i)/n$. Transforming the posterior density of ϕ to find the posterior density of $n^{\frac{1}{2}}\theta$, we are then able to calculate the sequence $(F_1(\xi,r_1), F_2(\xi,r_2),\cdots)$ of posterior distributions. Note that r_n enters F_n both from the posterior distribution of ϕ and through the standardizing quantities $\hat{\phi}$ and b.

We now state the main theorem of this chapter.

Theorem 3.1.1. If $\rho(\cdot)$ is analytic in some neighborhood of ϕ_o and $\rho(\phi_o)>0$, then there exist functions $\left\{\gamma_j(\xi,n)\right\}_{j=1}^{\infty}$ and for each integer N, there exist constants A and N_x such that

$$\left|F_n(\xi,r) - \Phi(\xi) - \sum_{j=1}^{N} \gamma_j(\xi,r)n^{-j/2}\right| \leq An^{-(N+1)/2} \text{ for all } n>N_x$$

on an almost sure set where the measure is generated by $\prod p_{\phi_o}(x_k)$. Here A depends on N, and N_x depends on N and the particular sequence $x = (x_1,x_2,\cdots)$.

The first two terms $\gamma_1(\xi,r)$ and $\gamma_2(\xi,r)$ are given explicitly below by (3.4.1) and (3.4.2) respectively. For a discussion of this theorem, see the end of Section 3.3.

3.2. Preliminaries.

Motivated by expression (3.0.3), we define a function $f_1(\cdot,\cdot)$ by

(3.2.1) $$f_1(\phi,r) = C(\phi)e^{\phi r}$$

for $\phi \in I$, the natural parameter space, and for r real. Define

(3.2.2) $$r_o = E_{\phi_o} R$$

Lemma 3.2.1. There exists a $d_1>0$ such that for fixed r with $|r-r_o| \leq d_1$, $f_1(\cdot,r)$ has a unique maximum at $\hat{\phi}(r)$ where $\hat{\phi}(r)$ satisfies

(3.2.3) $$C'(\hat{\phi})/C(\hat{\phi}) + r = 0$$

and $f_1(\phi, r)$ strictly decreases as ϕ moves away from $\hat{\phi}$ for all ϕ in I.

<u>Proof</u>: Lehmann (1959), Section 2.7, has shown that $C(\cdot)$ is analytic in a neighborhood of the complex variable $\phi + i\eta$ when ϕ is interior to the natural parameter space I and that the derivatives of $1/C(\phi)$ may be obtained by differentiating under the integral sign. Recall that $C(\phi)$ is defined by

(3.2.4) $$1/C(\phi) = \int e^{\phi R(x)} d\mu(x) \quad \text{for } \phi \in I.$$

Note that since ϕ is a real valued parameter and $R(x)$ is real valued, we have $0 < 1/C(\phi)$ all $\phi \in I$.

Differentiating (3.2.4), we find that

$$-C'(\phi)/C(\phi) = C(\phi) \int R(x) e^{\phi R(x)} d\mu(x)$$

or

(3.2.5) $$-C'(\phi)/C(\phi) = E_\phi R$$

and differentiating again gives

(3.2.6) $$\left[2C'(\phi)^2 - C''(\phi)C(\phi)\right]/C^2(\phi) = E_\phi R^2 \quad \text{for } \phi \text{ interior to I.}$$

It follows that

(3.2.7) $$\mathrm{Var}_\phi(R) = \left[C'(\phi)^2 - C''(\phi)C(\phi)\right]/C^2(\phi) \quad \text{for } \phi \text{ interior to I}$$

where $0 < \mathrm{Var}_\phi(R)$ since $R \neq$ constant, a.s. μ. Using the notation r_o from (3.2.2) and recalling that ϕ_o is interior to I, we have by (3.2.5)

(3.2.8) $$C'(\phi_o)/C(\phi_o) + r_o = 0 \quad .$$

Let $B(\phi) = C'(\phi)/C(\phi)$ for ϕ interior to I. Therefore $B(\phi_o) + r_o = 0$ and $B'(\phi) = -\mathrm{Var}_\phi(R)$. This last result follows from (3.2.7), and it is now clear that $B'(\phi) < 0$. Since ϕ_o is interior, it is a positive distance 2d from the boundary ∂I of I. That is

(3.2.9) $\qquad \operatorname{dist}(\phi_o, \partial I) = 2d > 0.$

Now $B(\cdot)$ is strictly monotone on $[\phi_o - d, \phi_o + d]$ so by the properties of the inverse function, (see Olmstead (1956), page 88) there exists an interval about r_o, say $|r-r_o| \le d_1$ where $\hat{\phi}(r)$ is the unique solution of (3.2.3) in the interval about ϕ_o. But $B(\cdot)$ is monotone on the interior of I so $\hat{\phi}(r)$ is the unique solution of (3.2.3).

Let r be fixed with $|r-r_o| \le d_1$. Now $f_1(\cdot,r)$ has derivative

(3.2.10) $\qquad [C'(\phi)/C(\phi) + r]C(\phi)e^{\phi r}$ for ϕ interior to I.

By (3.2.3), this vanishes at $\phi = \hat{\phi}(r)$ and since $C'(\phi)/C(\phi) = B(\phi)$ is strictly monotone, we have $f_1'(\phi,r) \gtrless 0$ as $\phi \lessgtr \hat{\phi}(r)$. The last assertion of the lemma is now immediate. Also by the one-sided continuity of $f_1(\cdot,r)$ at the endpoints, we see that $f_1(\cdot,r)$ has a unique maximum at $\hat{\phi}(r)$. Q.E.D.

Lemma 3.2.2. Let $\hat{\phi}(r)$ be the solution of (3.2.3). Then $\hat{\phi}(r)$ is a continuous function of r for $|r-r_o| \le d_1$.

Proof: The result follows from the inversion theorem mentioned in the previous proof.

Lemma 3.2.3. With r_o defined by (3.2.2) and $\hat{\phi}(r)$ given by (3.2.3), we have $\hat{\phi}(r_o) = \phi_o$.

Proof: By (3.2.5), we have

$$C'(\phi_o)/C(\phi_o) + r_o = 0,$$

but since $\hat{\phi}(r_o)$ is the unique solution of this equation, we must have $\phi_o = \hat{\phi}(r_o)$.

From the definition of Θ given by (3.1.1), we see that the posterior density of Θ given $\sum_{i=1}^{n} R(x_i)/n = r$ is proportional to

$$(3.2.11) \qquad \rho(\theta/b(r) + \hat{\phi}(r))\left[f_1(\theta/b(r) + \hat{\phi}(r),r)/f_1(\hat{\phi}(r),r)\right]^n$$

where $f_1(\cdot,\cdot)$ is given by (3.2.1), $\hat{\phi}(r)$ by (3.2.3) and

$$(3.2.12) \qquad b^2(r) = -f_1''(\hat{\phi}(r),r)/f_1(\hat{\phi}(r),r)$$

where prime denotes the derivative with respect to ϕ. This expression for $b(r)$ is equivalent to (3.1.2) as can be seen by direct substitution.

<u>Lemma 3.2.4</u>. Let $b(r)$ be defined by (3.2.12). Then $b(r)$ is a continuous function of r for $|r-r_o| \leq d_1$ and does not equal zero.

<u>Proof</u>: By Lemma 3.2.3, $\hat{\phi}(r)$ is a continuous function of r, and the result follows from (3.1.2) by composition of continuous functions since $C(\phi) > 0$. Now by (3.2.7), $b^2(r) = \mathrm{Var}_{\hat{\phi}}(R)$, and this is strictly greater than zero.

<u>Lemma 3.2.5</u>. Let $\rho(\cdot)$ be analytic in some neighborhood of ϕ_o. Then there exist positive constants η, δ_o ($\delta_o<1$), and d_o ($d_o<1$) such that $C(\theta/b(r) + \hat{\phi}(r))$ and $\rho(\theta/b(r) + \hat{\phi}(r))$ are analytic functions of θ for $|\theta| \leq \delta_o$ when r is fixed with $|r-r_o| \leq d_o$ and for these values $\mathcal{R}e\, C(\theta/b(r) + \hat{\phi}(r)) \geq \eta$.

<u>Proof</u>: Since $C(\cdot)$ is a continuous function, there exists a $d_3 > 0$ such that $|C(\phi)-C(\phi_o)| < C(\phi_o)/2$ for $|\phi-\phi_o| \leq d_3$. Take $\eta = C(\phi_o)/2$. Using the result below in (3.2.17), we find that the real part is greater than η. By the hypothesis of the theorem, $\rho(\phi)$ is analytic for $|\phi-\phi_o| \leq d_2$ for some $d_2 > 0$. Let $\varepsilon_1 = \min(d,d_2,d_3)$ where d is determined by (3.2.9). By Lemma 3.2.2 there exists a d_o with $d_o < \min(d_1,1)$ such that

$$(3.2.13) \qquad |\hat{\phi}(r) - \hat{\phi}(r_o)| \leq \varepsilon_1/3 \quad \text{for} \quad |r-r_o| \leq d_o \quad .$$

Lemma 3.2.3 showed that $\hat{\phi}(r_o) = \phi_o$. Introducing the notation

$$N_o = \{r: |r-r_o| \le d_o\} \tag{3.2.14}$$

for the closed interval around r_o, consider

$$\varepsilon_2 = \inf_{N_o} b(r) \quad . \tag{3.2.15}$$

Now $b(r)$ is continuous on N_o so that the minimum is attained. However by Lemma 3.2.4, this can not be zero. Therefore if δ_o satisfies

$$\delta_o < \min(\varepsilon_2 \cdot \varepsilon_1/3, 1), \tag{3.2.16}$$

we have

$$|\theta/b + \hat{\phi}-\phi_o| \le \varepsilon_1/3 + \varepsilon_1/3 = 2\varepsilon_1/3 \quad \text{if} \quad |\theta| \le \delta_o \text{ and } r \in N_o. \tag{3.2.17}$$

That is $\theta/b + \hat{\phi}$ lies in a closed interval of the natural parameter space, so by Lehmann (1959), Theorem 9, p. 52, $1/C(\theta/b + \hat{\phi})$ is analytic. Noting that $\rho(\cdot)$ is also analytic for such θ, the lemma is proved.

3.3. Expansion of the Posterior Distribution.

In this section, we obtain bounds similar to those in Chapter 1 which hold uniformly for r in some neighborhood N_o of r_o where N_o is given by (3.2.13). The previous argument is then modified to provide a proof for Theorem 3.1.1.

Define the function $f(\theta,r)$ by

$$f(\theta,r) = f_1(\theta/b + \hat{\phi},r)/f_1(\hat{\phi},r) \tag{3.3.1}$$

where $\hat{\phi} = \hat{\phi}(r)$ and $b = b(r)$ are defined by (3.2.3) and (3.1.2) respectively. The function $f_1(\cdot,\cdot)$ is defined in (3.2.1). Note that for fixed r, we have $f(0,r) = 1$, $f'(0,r) = 0$, and $f''(0,r) = -1$ where prime denotes the derivative with respect to θ. By (3.2.11), the posterior density of θ given r is proportional to

$$\rho(\theta/b + \hat{\phi})f^n(\theta,r) \quad . \tag{3.3.2}$$

Recall that θ is related to ϕ by (3.1.1).

We now establish some lemmas which ultimately lead to a proof of Theorem 3.1.1.

<u>Lemma 3.3.1</u>. Let $\rho(\phi)$ be analytic in some neighborhood of ϕ_o and let $f(\theta,r)$ be defined by (3.3.1). Then there exist a $\delta_2 > 0$, a sequence of functions $\left\{c_{\ell m}(r)\right\}_{\ell,m=0}^{\infty}$ and for each integer N, constants A_1 and A_2 depending on N such that

$$\left|\rho(\theta/b + \hat{\phi})f^n(\theta,r) - e^{-n\theta^2/2}\sum_{\ell+m\le N} c_{\ell m}(r)\,(\theta^3 n)^{\ell}\theta^m\right|$$
$$\le e^{-n\theta^2/2}\left[A_1|\theta|^{N+1} + A_2|\theta^3 n|^{N+1}\right] \quad \text{all } |\theta|\le\delta \text{ and } |n\,\theta^3|\le 1$$

for r fixed and $r\in N_o$ where N_o is defined by (3.2.13). The constants A_1 and A_2 do not depend on r for $r\in N_o$.

<u>Proof</u>: Let $r\in N_o$ be fixed temporarily. By Lemma 3.2.5 and the definition of N_o, $\rho(\theta/b + \hat{\phi})$ and $C(\theta/b + \hat{\phi})$ are analytic functions of θ for $|\theta| \le \delta_o$. Also $\mathcal{R}e\, C(\theta/b + \hat{\phi}) \ge \eta > 0$ for some η, so that $C(\theta/b + \hat{\phi})$ does not equal zero. Define $h(\theta,r)$ by

$$(3.3.3)\qquad h(\theta,r) = \log f(\theta,r)$$
$$= \log\left[\exp\left[\theta r/b\right] C(\theta/b + \hat{\phi})/C(\hat{\phi})\right] \quad .$$

Now $f(0,r) = 1$ so that there is some neighborhood of zero, perhaps depending on r, where the principal branch of the logarithm is an analytic function. We now show that this neighborhood is independent of r for $r\in N_o$ and that $h(\theta,r)$ may be expressed as

$$(3.3.4)\qquad h(\theta,r) = \log\left[C(\theta/b + \hat{\phi})/C(\hat{\phi})\right] + \theta r/b$$

where the logarithm is again the principal value. For $r\in N_o$, we have $|\hat{\phi}-\phi_o| < \varepsilon_1/3$ by (3.2.13). Recalling the steps leading up to this expression, we see that this implies that $|C(\hat{\phi})-C(\phi_o)| < \eta$ or $\eta < C(\hat{\phi}) < 3\eta$ since $C(\hat{\phi})$ is real and positive. Therefore

$\mathcal{Re}\left[C(\theta/b + \hat{\phi})/C(\hat{\phi})\right] \geq \eta/3\eta = 1/3$ for $|\theta| \leq \delta_0$ and all $r \in N_0$. Thus the argument of the ratio lies between $-\pi/2$ and $\pi/2$. Also $|r/b| < 1/\varepsilon_2$ by (3.2.15) so that by decreasing δ_0 to some δ_1 if necessary, we have $|\theta r/b| \leq \pi/4$ for $|\theta| \leq \delta_1$ all $r \in N_0$. Clearly the expression (3.3.4) holds for this range of values. We may also write

$$(3.3.5) \qquad h(\theta, r) = \log C(\theta/b + \hat{\phi}) - \log C(\hat{\phi}) + \theta r/b \quad .$$

Consider then the bound

$$\left|\log C(\theta/b + \hat{\phi})\right| \leq \sup_{|\phi-\phi_0| \leq 2\varepsilon_1/3} \left|\log C(\phi)\right| \quad \text{for} \quad |\theta| \leq \delta_1 \text{ and } r \in N_0$$

which follows from (3.2.17), since $\delta_1 < \delta_0$. Here ε_1 is defined in the proof of Lemma 3.2.5. Using the bounds on $C(\hat{\phi})$ and $\theta r/b$ obtained above, we see that there exists an M such that

$$(3.3.6) \qquad |h(\theta, r)| \leq M < \infty \quad \text{for} \quad |\theta| \leq \delta_1 \text{ and } r \in N_0 \quad .$$

For fixed $r \in N_0$, $h(\theta, r)$ is analytic for $|\theta| \leq \delta_1$ and its derivatives are given by the Cauchy formula. More precisely,

$$(3.3.7) \qquad h^{(s)}(0,r)/s! = \frac{1}{2\pi i} \int_{\Gamma} \frac{h(t,r)}{t^{s+1}} dt \quad \text{for} \quad s=1,2,\cdots$$

where $\Gamma = \{t: |t-0| = 2\delta_1/3\}$. We then estimate

$$(3.3.8) \qquad |h^{(s)}(0,r)/s!| \leq M(2\delta_1/3)^{-s} \text{ for } s=1,2,\cdots, \text{ and for all } r \in N_0 \ .$$

Using the Taylor expansion and substituting the known values of $f(0,r)$, $f'(0,r)$, and $f''(0,r)$, we have

$$(3.3.9) \qquad nh(\theta, r) = -n\theta^2/2 + (n\theta^3) \sum_{s=3}^{\infty} a_s(r)\theta^{s-3}$$

for $|\theta| \leq \delta_1$ and fixed $r \in N_0$. The coefficients are given by the relation

$$(3.3.10) \qquad a_s(r) = h^{(s)}(0,r)/s! \quad \text{for} \quad s=3,4,\cdots$$

or equivalently, by using the right hand side of (3.3.7).

Let $w = n\theta^3$ and

$$(3.3.11) \qquad \psi(\theta,r) = \sum_{s=3}^{\infty} a_s(r)\theta^{s-3}$$

so that

$$(3.3.12) \qquad \rho(\theta/b + \phi)f^n(\theta,r) = \rho(\theta/b + \phi)e^{-n\theta^2/2}\, e^{w\psi(\theta,r)}$$

for $|w| \le 3$ and $|\theta| \le \delta_1$.

Bounding the coefficients $a_s(r)$ by (3.3.8), it follows that

$$|\psi(\theta,r)| \le 27M/8\,\delta_1^3 \sum_{s=3}^{\infty} |3\theta/2\,\delta_1|^{s-3} \text{ for } |\theta| \le \delta_1/2 \text{ and } r \in N_o$$

or

$$(3.3.13) \qquad |\psi(\theta,r)| \le M_1 < \infty \text{ for } |\theta| \le \delta_1/2 \text{ and } r \in N_o.$$

Consider then the function $P(w,z,r)$ defined by

$$(3.3.14) \qquad P(w,z,r) = \rho(z/b + \phi)e^{w\psi(z,r)} \text{ for } |z| \le \delta_1,\ |w| \le 3, \text{ and } r \in N_o.$$

Now $P(w,z,r)$ is an analytic function for $|z| \le \delta_1$ and $|w| \le 3$ for each fixed $r \in N_o$. Bounding $\rho(z/b + \phi)$ separately and then using (3.3.13), we have

$$(3.3.15) \qquad |P(w,z,r)| \le M_2 < \infty \text{ for } |z| \le \delta_1/2,\ |w| \le 3, \text{ all } r \in N_o.$$

Also for each fixed $r \in N_o$, we have the expansion

$$(3.3.16) \qquad P(w,z,r) = \sum_{\ell,m} c_{\ell m}(r) w^{\ell} z^{m}$$

where the coefficients are given by the Cauchy integral formula

$$(3.3.17) \qquad c_{\ell m}(r) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1 \times \Gamma_2} \frac{P(w,z,r)}{w^{\ell+1} z^{m+1}}\, dw\, dz$$

where $\Gamma_1 = \{w: |w| = 2\}$ and $\Gamma_2 = \{z: |z| = \delta_1/3\}$ (see Fuks (1963), pages 39-40 or Markushevich (1965), pages 101-105). Using the usual estimates for the integral, we find that

$$(3.3.18) \qquad |c_{\ell m}(r)| \le M_2(\delta_1/3)^{-m} 2^{-\ell} \text{ for } \ell, m = 0,1,\cdots,$$

where M_2 is given by (3.3.15). Therefore for every integer N, we have

$$\left|\sum_{\ell+m>N} c_{\ell m}(r)w^{\ell}z^{m}\right| \le M_2 \sum_{\ell+m>N} |w/2|^{\ell}\,|3z/\delta_1|^{m} \le A_1|w|^{N+1} + A_2|z|^{N+1}$$

for $|w| \le 1$ and $|z| \le \delta_1/4$ all $r \in N_0$. This last inequality follows from the same argument that leads to (1.4.10). Let $\delta_2 = \delta_1/4$ and the lemma follows.

<u>Lemma 3.3.2</u>. Let $f(\theta,r)$ be defined by (3.3.1). Then there exists a $\delta_3 > 0$ depending on N_0 such that

$$\log f(\theta,r) \le -\theta^2/4 \quad \text{all real } \theta \text{ with } |\theta| \le \delta_3 \text{ and } r \in N_0.$$

<u>Proof</u>: By (3.3.9) and (3.3.11), we have

$$\log f(\theta,r) = -\theta^2/2 + \theta^3\psi(\theta,r) \quad \text{for } |\theta| \le \delta_1 \text{ and } r \in N_0.$$

The existence of a $\delta_3 > 0$ follows from the bound (3.3.13) on $|\psi(\theta,r)|$.

Note that the last two lemmas remain true if δ_3 and δ_4 are replaced by δ where

$$\delta = \min(\delta_3, \delta_4) . \tag{3.3.19}$$

<u>Lemma 3.3.3</u>. Let $f(\theta,r)$ be defined by (3.3.1). Then there exists an α with $0<\alpha<1$ such that

$$f(\theta,r) \le 1-\alpha \quad \text{all real } \theta \text{ with } |\theta| \ge \delta \text{ and } r \in N_0 .$$

Here δ is given by (3.3.19).

<u>Proof</u>: For fixed r, Lemma 3.2.1 tells us that $f_1(\phi,r)$ strictly decreases as ϕ moves away from ϕ. Recalling (3.3.1), we see that

$$f(\theta,r) \le \max\left[f(-\delta,r), f(\delta,r)\right] \quad \text{for } r \in N_0. \tag{3.3.20}$$

The right hand side is bounded by $e^{-\delta^2/4}$ for all $r \in N_0$ according to Lemma 3.3.2. This lemma follows with $\alpha = 1-e^{-\delta^2/4}$.

Since the first part of the proof of Theorem 3.3.1 parallels the work of Chapter 1, we only sketch the details.

Proof of Theorem 3.1.1: Let N be arbitrary but fixed. Let δ be given by (3.3.19). By Lemma 3.3.3, we have

$$\left\{\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right\}\rho(\theta/b + \tilde{\emptyset})f^n(\theta,r)d\theta \le (1-\alpha)^n \int_{-\infty}^{\infty}\rho(\theta/b + \tilde{\emptyset})d\theta$$

$$\le (1-\alpha)^n\cdot b(r)$$

$$\le M(1-\alpha)^n \quad \text{all } n \text{ for } r\in N_o \tag{3.3.21}$$

some M since $b(r)$ is bounded according to Lemma 3.2.4 and (3.2.14).

Repeating the proof of Lemma 1.4.1 with the bound from Lemma 3.3.2, we find that there exists an M_1 such that

$$\left\{\int_{-\delta}^{-n^{-1/3}} + \int_{n^{-1/3}}^{\delta}\right\}\rho(\theta/b + \tilde{\emptyset})f^n(\theta,r)d\theta \le M_1\exp\left[n^{1/3}/4\right] \quad (n>\delta^{-3}) \tag{3.3.22}$$

for $r\in N_o$.

Denote by $P_N(w,z,r)$ the truncated series $\sum_{\ell+m\le N} c_{\ell m}(r)w^\ell z^m$ of $P(w,z,r)$. Using Lemma 3.3.1 together with the inequalities (3.3.21) and (3.3.22), we proceed as in Chapter 1 from Lemma 1.4.6 through Theorem 1.5.2. We then obtain for each integer N the estimates

$$\left|\int_{-\infty}^{\infty}\rho(\theta/b + \tilde{\emptyset})f^n(\theta,r)d\theta - \int_{-\infty}^{\infty}e^{-n\theta^2/2}P_N(n\theta^3,\theta,r)d\theta\right|$$

$$\le B_1 n^{-(N+1)/2} \quad \text{all } n>N_{B_1} \tag{3.3.23}$$

and

$$\left|\int_{-\infty}^{\xi n^{-1/2}}\rho(\theta/b + \tilde{\emptyset})f^n(\theta,r)d\theta - \int_{-\infty}^{\xi n^{-1/2}}e^{-n\theta^2/2}P_N(\theta^3 n,\theta,r)d\theta\right|$$

$$\le B_2 n^{-(N+1)/2} \quad \text{all } n>N_{B_2} \tag{3.3.24}$$

some B_1, B_2, N_{B_1}, and N_{B_2} for all $r\in N_o$. This last expression will be uniform in ξ as in the proof of Theorem 1.5.1. We then divide the expansions as in Section 1.5. That is, there exist an A independent of $r\in N_o$ and N_A depending on A, such that

$$\left|F_n(\xi,r) - \Phi(\xi) - \sum_{j=1}^{N} \gamma_j(\xi,r)n^{-j/2}\right| \le An^{-(N+1)/2} \quad (n > N_A) \quad .$$

Here $\gamma_j(\xi,r)$ is determined in the same way as the $\gamma_j(\xi)$ of Theorem 1.5.3.

Now consider the stochastic aspect of the problem. We have $E_{\phi_o} R = r_o < \infty$ so that by the Strong Law of Large Numbers, $\bar{R} = \sum_{i=1}^{n} R(x_i)/n \xrightarrow{a.s.} r_o$ on the product space having measure P_{ϕ_o} induced by $\prod_{i=1}^{\infty} p_{\phi_o}(x_i)$, where p_{ϕ_o} is given by (3.0.1) with $\phi = \phi_o$. Therefore we have a set D with $P_{\phi_o}(D) = 1$ such that for every $x = (x_1, x_2, \cdots)$ with $x \in D$, there exists an N_x such that $|\bar{R} - r_o| \le \delta$ if $n > N_x$. Repeating the above argument for each $x \in D$, we complete the proof.

The conclusion of Theorem 3.1.1 states that, almost surely, the observed sequence $(x_1, x_2, \cdots)$ provides a sequence of values of r for which the asymptotic expansion of $F_n(\xi,r)$ is valid. That is, the extra terms may be used as correction terms when n is sufficiently large. Using the notation of Mann and Wald (1943), we could state the conclusion in the weaker form

$$(3.3.25) \qquad F_n(\xi,r) - \Phi(\xi) - \sum_{j=1}^{N} n^{-j/2} \gamma_j(\xi,r)n^{-j/2} = O_p(n^{-(N+1)/2}) \quad (n \to \infty) \ .$$

However if it is the case that there is only convergence in probability, it could happen that $F_n(\xi,r)$ would not have an asymptotic expansion or even a limiting distribution for any sequence $(x_1, x_2, \cdots)$. For our conclusion then, it may be appropriate to introduce the notation $O_{a.s.}(n^{-(N+1)/2})$ analogous to Mann and Wald's O_p notation. We do not do so however and only remark that the operational calculus of such functions would be, up to a finite number of operations, the same as the ordinary relationships.

It was decided to center at the maximum likelihood estimate $\hat{\phi}$ after first attempting to center at the true value ϕ_o. The first method led to difficulties with the expansion of $\log f(\theta,r)$, and even the convergence to the normal distribution may not be true. As to the choice of scaling constant given in (3.1.2), there is a question as to whether to use the Fisher information evaluated at ϕ_o or at $\hat{\phi}$ or possibly some other value. To show convergence to the normal distribution, either of the first two may be used. LeCam (1953) used ϕ_o and in LeCam (1958) $\hat{\phi}$ was used to show the convergence for quite general likelihoods. LeCam (1953), Theorem 7, and (1958), Lemma 5, proved that the posterior distribution converges "in variation" almost surely. It is interesting to compare the conclusion of Theorem 3.1.1 with the results of LeCam specialized to the density given in (3.0.1). In this special case, our work leads to the same conclusion of convergence "in variation". This is shown in Appendix A.3.

3.4. Calculation of Terms and Examples.

The results of Section 1.6 may be used directly giving γ_1 and γ_2 in terms of $c_{\ell m}$. However, it must be remembered that the $c_{\ell m}$ given by (3.3.15) are functions of r. In particular, we have from (1.6.4) and (1.6.5)

$$\gamma_1(\xi,r) = -\varphi(\xi)c_{00}^{-1}(r)\left[c_{10}(r)(\xi^2+2) + c_{01}(r)\right] \tag{3.4.1}$$

and

$$\gamma_2(\xi,r) = -\varphi(\xi)c_{00}^{-1}(r)\Big[c_{20}(r)\xi^5 + (5c_{20}(r) + c_{11}(r))\xi^3 + (15c_{20}(r) + 3c_{11}(r) + c_{02}(r))\xi\Big]. \tag{3.4.2}$$

To evaluate these expressions, we consult Table 1.6.1 which enables us to express $c_{\ell m}(r)$ in terms of $\rho(\theta/b + \hat{\phi})$ and $\log f(\theta,r)$ where $f(\theta,r)$ is defined in (3.3.1). From the definition of $f(\theta,r)$, we see that the $a_s(r)$ defined by (3.3.5) is also equal to

$$\frac{d^s \log C(\phi)}{d\phi^s} \Big/ b(r)^{s/2} s! \tag{3.4.3}$$

for $s \geq 3$. Here $b(r)$ is given by (3.1.2) in terms of $C(\cdot)$. The above steps lead to the following expressions for the $c_{\ell m}$ which in turn enable us to express $\gamma_1(\xi,r)$ and $\gamma_2(\xi,r)$ in terms of $\rho(\cdot)$ and $C(\cdot)$ and their derivatives. Define $B = (C'^2 - CC'')^{-\frac{1}{2}}$

$$\left\{\begin{aligned}
c_{00}(r) &= \rho \\
c_{10}(r) &= \rho\left[C^2C''' - 3C''C'C + 2C'^3\right]B^3/6 \\
c_{01}(r) &= \rho' CB \\
c_{20}(r) &= \rho\left[C^2C''' - 3C''C'C + 2C'^3\right]^2 B^6/72 \\
c_{11}(r) &= B^4\Big\{\rho'\left[C^3C''' - 3C''C'C^2 + 2C'^3C\right]/6 \\
&\qquad + \rho\left[C''''C^3 - 4C'''C'C^2 - 3C''^2C^2 + 12C'^2C''C - 6C'^4\right]\Big\} \\
c_{02}(r) &= \rho'' C^2 B^2/2
\end{aligned}\right. \tag{3.4.4}$$

Here $B(\cdot)$, the prior density $\rho(\cdot)$, and the normalization constant for the exponential family $C(\cdot)$, together with all their derivatives, are evaluated at $\hat{\phi}(r)$ where $\hat{\phi}(r)$ is the solution of (3.2.3).

The manner in which the prior density enters the asymptotic expansions of Theorem 3.1.1 is now apparent. In the term of order $n^{-\frac{1}{2}}$, it enters only as $\rho'(\hat{\phi})/\rho(\hat{\phi})$, and in the term of order n^{-1}, it appears as $\rho''(\hat{\phi})/\rho(\hat{\phi})$ and as $\rho'(\hat{\phi})/\rho(\hat{\phi})$ when $c_{11}(r) \neq 0$. It is shown below that $c_{11}(r) = 0$ for a normal likelihood.

In situations where the prior distribution is constant in some neighborhood of ϕ_0, the prior distribution will not enter into the expansion of order n^{-1}.

We now consider a few examples where $p_\phi(x)$ is given by (3.0.1) and the prior density $\rho(\cdot)$ satisfies the assumptions of Theorem 3.1.1. The first is the Bernoulli case where μ is counting measure and $R(x)$

is zero or one. In this case $\phi = \log[p/(1-p)]$ and $C(\phi) = (1 + e^{\phi})^{-1}$. The transformed variable θ is equal to $[\phi - \log[r/(1-r)]][r(1-r)]^{\frac{1}{2}}$.

Upon calculating the derivatives of $C(\cdot)$ and substituting into the expressions above, we find that

$$\gamma_1(\xi,r) = -\varphi(\xi)\left[(2r-1)(\xi^2+2)/6 + \rho'(r)/\rho(r)\right][r(1-r)]^{-\frac{1}{2}}$$

and

$$\gamma_2(\xi,r) = -\varphi(\xi)[r(1-r)]^{-1}\Big\{(2r-1)^2\xi^5/72 - (4r^2-9r+4)\xi^3/36 - (4r^2-9r+4)\xi/12 + (2r-1)\rho'(r)\xi^3/6\rho(r) + \left[(2r-1)\rho'(r) + \rho''(r)\right]/2\rho(r)\Big\}.$$

The above result is not for the usual parameter. The fact that the posterior distribution of $n^{\frac{1}{2}}(p-r)/[r(1-r)]^{\frac{1}{2}}$ does converge to the standard normal distribution with probability one follows easily since p is obtained as a smooth transformation of ϕ. This result should be compared with von Mises (1964), pages 345-347 and Bernstein (1934), page 406 who consider $n^{\frac{1}{2}}(p-r)/[r(1-r)]^{\frac{1}{2}}$ and take limits while ignoring the stochastic aspect of r.

As a second example, we consider the normal distribution with known variance σ_o^2 and mean m. Here $p_\phi(x)$ has $\phi = m/\sigma_o^2$, $C(\phi) = \exp[-\sigma_o^2\,\phi^2/2]$ and $R(x) = x$. The transformed variable θ equals $[\phi - \bar{x}/\sigma_o^2]\,\sigma_o$ which can be written as $(m-\bar{x})/\sigma_o$.

The calculations could be done directly by finding the derivatives of $C(\phi)$. However in this example, it is easier to find $a_3(r)$ and $a_4(r)$ by (3.4.3) and then to use Table 1.6.1 to calculate $\gamma_1(\xi,r)$ and $\gamma_2(\xi,r)$. In particular we find that

$$\gamma_1(\xi,r) = -\varphi(\xi)\rho'(\bar{x})/\sigma_o\rho(\bar{x})$$

and

$$\gamma_2(\xi,r) = -\varphi(\xi)\rho''(\bar{x})/\sigma_o^2\,\rho(\bar{x}) \quad .$$

If the prior density is constant in some neighborhood of ϕ_o, the c.d.f.

of $n^{\frac{1}{2}}\theta$ is normal up to terms of order $n^{-3/2}$. Theorem 3.1.1 gives an expansion which may be compared to Gnedenko (1962), page 414, second equation. Gnedenko ignores the stochastic aspect of $\bar{x}$ and does not give any terms beyond the limit function $\Phi(\cdot)$. On the previous pages, he obtains an expansion for the density in terms of $\varphi(\cdot)$.

We conclude with some tabulated results for well known exponential distributions.

Table 3.4.1. First correction terms.

Density	ϕ	$\hat{\phi}(r)$	$b(r)$	$\mathcal{V}_1(\xi,r)$ *
Poisson $\lambda^x e^{-\lambda}/x!$ $x=0,1,2,\cdots$	$\log\lambda$	$\log\bar{x}$	$\bar{x}^{\frac{1}{2}}$	$\varphi(\xi)\left[(\xi^2+2)/6+\rho'/\rho\right](\bar{x})^{-\frac{1}{2}}$
Gamma: α known $\lambda^\alpha e^{-\lambda x}x^{\alpha-1}/\Gamma(\alpha)$ $x>0$	λ	$\alpha/\bar{x}$	$\bar{x}\alpha^{-\frac{1}{2}}$	$-\varphi(\xi)\left[(\xi^2+2)/3\alpha^{\frac{1}{2}}+(\rho'/\rho)(\alpha^{\frac{1}{2}}/\bar{x})\right]$
Normal: mean m_0 known $(2\pi\sigma^2)^{-\frac{1}{2}}e^{-(x-m_0)^2/2\sigma^2}$ $-\infty<x<\infty$	$1/\sigma^2$	$n/\sum_{i=1}^{n}(x_i-m_0)^2$	$\sum_{i=1}^{n}(x_i-m_0)^2/2^{\frac{1}{2}}n$	$-\varphi(\xi)\left\{2^{\frac{1}{2}}(\xi^2+2)/3 + \rho'/\rho\left[2^{\frac{1}{2}}n/\sum_{i=1}^{n}(x_i-m_0)^2\right]\right\}$
Normal: variance σ_0^2 known $(2\pi\sigma_0^2)^{-\frac{1}{2}}e^{-(x-m)^2/2\sigma_0^2}$ $-\infty<x<\infty$	m/σ_0^2	$\bar{x}/\sigma_0^2$	σ_0	$-\varphi(\xi)\rho'/\sigma_0\rho$
Bernoulli $p^x(1-p)^{1-x}$ $x=0,1$	$\log[p/(1-p)]$	$\log[r/(1-r)]$	$[r(1-r)]^{\frac{1}{2}}$	$-\varphi(\xi)\left[(2r-1)(\xi^2+2)/6+\rho'/\rho\right][r(1-r)]^{-\frac{1}{2}}$

* Both $\rho(\cdot)$ and $\rho'(\cdot)$ are evaluated at $\hat{\phi}(r)$.

Appendix A.1. Background of Asymptotic Theory.

It seems advisable to state the basic definitions connected with asymptotic expansions. The notation used here is similar to that of de Bruijn (1961).

Throughout this work, n takes on only integer values so that the definitions are stated for that particular case.

The notation $\psi_1(n) = O(\psi_2(n))$ $(n \to \infty)$ means that there exist constants m and M such that

(a.1.1) $$|\psi_1(n)| \leq M\,|\psi_2(n)| \text{ whenever } m < n < \infty .$$

This notation is sometimes modified to read $\psi_1(n) = O(\psi_2(n))$ $(n>m)$. When $|\psi_1(n)/\psi_2(n)| \neq \infty$ (all n) and when (a.1.1) holds for sufficiently large n, we may extend it to hold for all integer n greater than unity. With functions involving an additional variable ξ, we have the notion of a bounding constant which is the same for all ξ. The O-symbol is uniform with respect to ξ if there exist constants m and M such that

(a.1.2) $$|\psi_1(n,\xi)| \leq M\,|\psi_2(n,\xi)| \quad \text{whenever} \quad m < n < \infty ,$$

where m and M can be chosen to be independent of ξ.

The notation $\psi_1(n) = o(\psi_2(n))$ $(n \to \infty)$ means that $\psi_1(n)/\psi_2(n)$ tends to zero as n goes to infinity.

Suppose we have a sequence of functions $\psi_0(n), \psi_1(n), \cdots$, satisfying

$$\psi_j(n) = o(\psi_{j-1}(n)) \quad (n \to \infty)$$

each $j=1,2,\cdots$ and there exists a sequence of constants $c_0, c_1, c_2, \cdots$ such that the following sequence of O-formulas holds for f(n).

$$
(a.1.3)\quad \begin{cases} f(n) = O(\psi_0(n)) & (n \to \infty) \\ f(n) = c_0\psi_0(n) + O(\psi_1(n)) & (n \to \infty) \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ f(n) = c_0\psi_0(n) + \cdots + c_{j-1}\psi_{j-1}(n) + O(\psi_j(n)) & (n \to \infty) \\ \cdots\cdots\cdots\cdots\cdots\cdots & \end{cases}
$$

The whole set of formulas is represented by the single formula

$$
(a.1.4)\qquad f(n) \sim c_0\,\psi_0(n) + \cdots + c_j\,\psi_j(n) + \cdots \qquad (n \to \infty)\ .
$$

The right hand side of (a.1.4) is called an asymptotic expansion for $f(n)$. We remark that $\sim$ is used in (a.1.4) since the right hand side may or may not converge.

When the function of interest depends on a second parameter ξ, we again define the series in terms of the sequence $\{\psi_j(\cdot)\}$. This time, however, the coefficients c_j are functions of ξ as are the O-functions in the set of equations (a.1.3). It is usually desirable, although not necessary, to have these O-functions uniform with respect to the parameter ξ.

Now the asymptotic expansion (a.1.4) may not be unique, for if there is a function $h(n)$, with $h(n) = o(\psi_j(n))$ $(n \to \infty)$ for all j,

$$
h(n) \sim c_0\,\psi_0(n) + \cdots + c_j\,\psi_j(n) + \cdots \qquad (n \to \infty)
$$

where $c_j=0$ (all j). It follows that if $f(n)$ is any function having an expansion, both $f(n)$ and $f(n) + h(n)$ have the same expansion. From this, we also see that if $f(n)$ is altered by subtracting a portion which is $o(\psi_j(n))$ for all j, the asymptotic expansion will remain unchanged.

Further information regarding asymptotic expansions in general may be obtained from Copson (1965), de Bruijn (1961), and Erdélyi (1956).

Appendix A.2. Examples Related to Wasow's Paper.

As mentioned in the introduction, Wasow has conclusions much like those of Chapter 1 and Chapter 2 of this thesis. Wasow's conditions (A) and (E) imply that for sufficiently large λ,

$$|g(t,\lambda)| \le M \quad \text{all } t \text{ for some } M \tag{a.2.1}$$

and

$$\int_{-\infty}^{\infty} g(t,\lambda)t^m dt < \infty \qquad m \ge 0 \text{ arbitrary.} \tag{a.2.2}$$

In our application, $g(t,\lambda)$ would correspond to the density of $n^{\frac{1}{2}}X_n$ with $\lambda = n^{\frac{1}{2}}$, so that his g is related to our k and f by

$$g(t,\lambda) = k(t/\lambda)f^{\lambda^2}(t/\lambda)\Big/\lambda \int_{-\infty}^{\infty} k(u)f^{\lambda^2}(u)du \tag{a.2.3}$$

where the denominator equals $(2\pi)^{\frac{1}{2}}k(0) + O(\lambda^{-1})$ as can be seen for example by Theorem 1.4.1. Consider first an example where $\sup_t g(t,\lambda) = \infty$ all λ. Let $f(x) = e^{-x^2/2}$ all x. Let

$$k(x) = \begin{cases} (1+x^2)^{-1} & , -\frac{1}{2} \le x \le \frac{1}{2} \\ e^{2j^3}2^j(x-j+e^{-j^3}2^{-j}) & , j-e^{-j^3}2^{-j} \le x \le j \quad \text{for } j=3,4,\cdots \\ -e^{2j^3}2^j(x-j)+e^{j^3} & , j < x \le j+e^{-j^3}2^{-j} \quad \text{for } j=3,4,\cdots \\ 0 & , \text{elsewhere} \quad . \end{cases}$$

In this case, it can be shown by straightforward calculations that (a.2.1) does not hold, but that the assumptions of our Theorem 1.5.3 are nevertheless satisfied.

As an example where none of the moments are finite so that (a.2.2) is violated, take $k(x) \equiv 1$ and

$$f(x) = \begin{cases} e^{-x^2/2} & , -1 \leq x \leq 1 \\ 1/\log(x+1) & , j \leq x \leq j + j^{-3/2} \quad \text{for } j=4,5,\cdots \\ 0 & , \text{elsewhere.} \end{cases}$$

In this case, our assumptions are clearly satisfied. While Wasow's results are more restrictive in these respects, they are less restrictive than ours in that the functions $g(t, \lambda)$ do not necessarily have the form (a.2.3).

Appendix A.3. Convergence of the Posterior Distribution in Variation.

LeCam (1953,1958) has shown that under quite general conditions, the posterior distribution converges "in variation" to the normal distribution. The metric of this convergence is $\|P-Q\| = 2 \sup_{B \in \mathcal{B}} |P(B)-Q(B)|$ where $\mathcal{B}$ is the Borel field. We now demonstrate that our expansions may be used to obtain the same result when the population has density (3.0.1).

It can be shown that

$$\text{(a.3.1)} \qquad \|P-Q\| = \int |p-q|\,dv$$

where $p = dP/dv$, $q = dQ/dv$, and v is some measure dominating both P and Q. In the present situation, the limiting distribution $\Phi(\cdot)$ has density $\phi(\cdot)$ with respect to Lebesgue measure and $\xi = n^{\frac{1}{2}}\theta$ also has a density. Now ξ has the distribution function

$$\text{(a.3.2)} \qquad F_n(\xi,r) = \int_{-\infty}^{\xi n^{-\frac{1}{2}}} \rho(\theta/b + \hat{\emptyset})f^n(\theta,r)d\theta \Big/ \int_{-\infty}^{\infty} \rho(\theta/b + \hat{\emptyset})f^n(\theta,r)d\theta$$

so that the density function $f_n(\xi,r)$ is given by

$$\text{(a.3.3)} \qquad f_n(\xi,r) = \rho(\hat{\emptyset} + \xi n^{-\frac{1}{2}}/b)f^n(\xi n^{-\frac{1}{2}},r)\Big/n^{\frac{1}{2}} \int_{-\infty}^{\infty} \rho(\theta/b + \hat{\emptyset})f^n(\theta,r)d\theta \ .$$

Again we argue pointwise on the almost sure set where $\Sigma\, R(x_i)/n$ converges to r_o. If the assumptions of Theorem 3.1.1 hold, the denominator of (a.3.3) equals

$$\text{(a.3.4)} \qquad \rho(\hat{\emptyset})(2\pi)^{\frac{1}{2}} + O(n^{-1}) \qquad (n \to \infty)$$

where the O-function is uniform for the values of r under consideration.

Now

$$(a.3.5) \qquad \|F_n - \Phi\| = \int_{-\infty}^{\infty} |\varphi(\xi) - f_n(\xi,r)|\,d\xi$$
$$\leq \int_{|\xi|\geq n^{1/6}} \varphi(\xi)\,d\xi + \int_{|\xi|\geq n^{1/6}} f_n(\xi,r)\,d\xi + \int_{|\xi|\leq n^{1/6}} |\varphi(\xi) - f_n(\xi,r)|\,d\xi$$
$$= I_1(n) + I_2(n) + I_3(n) \qquad \text{say.}$$

Here $I_1(n) \to 0$ by the dominated convergence theorem. In $I_2(n)$, let $u = \xi n^{-\frac{1}{2}}$ so that

$$I_2(n) = n^{\frac{1}{2}} \int_{|u|\geq n^{-1/3}} \rho(u/b + \hat{\emptyset}) f^n(u,r)\,du \Big/ \left[(2\pi)^{\frac{1}{2}}\rho(\hat{\emptyset}) + 0(n^{-1})\right]$$

which goes to zero by (3.3.21) and (3.3.22). We turn our attention to $I_3(n)$ and note that Lemma 3.3.1 with θ replaced by $\xi n^{-\frac{1}{2}}$ gives

$$(a.3.6) \quad |\rho(\hat{\emptyset} + \xi n^{-\frac{1}{2}}/b) f^n(\xi n^{-\frac{1}{2}}, r) - e^{-\xi^2/2}\rho(\hat{\emptyset})| \leq e^{-\xi^2/2}\left[A_1 |n^{-\frac{1}{2}}\xi| + A_2 |\xi^3 n^{-\frac{1}{2}}|\right]$$

for some A_1 and A_2 when $|\xi| \leq \delta n^{\frac{1}{2}}$ and $|\xi^3 n^{-\frac{1}{2}}| \leq 1$. That is, for sufficiently large n, the bound works for $|\xi| \leq n^{1/6}$. Dividing (a.3.6) by the denominator of f_n in (a.3.3) which equals $(2\pi)^{\frac{1}{2}}\rho(\hat{\emptyset}) + 0(n^{-1})$ and adding $|\varphi(\xi) - e^{-\xi^2/2}/\left[(2\pi)^{\frac{1}{2}} + 0(n^{-1})\right]|$ to the result, we obtain a bound for the integrand of $I_3(n)$. Integrating, we obtain a bound on $I_3(n)$ which goes to zero. Together with the previous results, this shows that $\|F_n - \Phi\|$ goes to zero. That is, on the almost sure set where $\Sigma R(x_i)/n$ converges to r_o, $F_n(\xi,r)$ converges in variation to $\Phi(\xi)$. LeCam (1958) actually makes the equivalent statement that the difference between the posterior distribution of $\emptyset$ and a normal distribution with mean $\hat{\emptyset}$ and variance $1/nb^2(r)$ goes to zero in variation. Thus we are able to obtain a conclusion similar to LeCam's under more restrictive assumptions (as mentioned in Section 3.3).

Appendix A.4. Relation to Laplace.

The method used to obtain the expansion in Chapter 1 of this thesis is originally due to Laplace (see Laplace (1847), Book One, Sections 22-27). We proved the existence of an expansion for the integral $\int k(x)f^n(x)dx$ by following the development given by de Bruijn (1961) in which a double series approximation is found for $k(x)f^n(x)$. An alternate method of giving a rigorous proof of Laplace's approximation would be to verify his original steps, which we now sketch. Consider for simplicity the case where $k(x) \equiv 1$. We assume as in Chapter 1 that $f(x)$ has a mode at $x=0$ and that $f(x)$ is analytic for $|x| \leq \delta$ for some $\delta > 0$. It will also be assumed that $f(0)=1$, $f'(0)=0$, and $f''(0)= -1$. Laplace uses the notation $y(x)=f^n(x)$, and then he defines a quantity $v = x(-\log y)^{-\frac{1}{2}}$ and denotes by U, $dU^2/dx, \ldots$ the values of v, $dv^2/dx, \ldots$ evaluated at $x=0$. Consider the integral of $y(x)$ from $-\theta$ to θ' where θ and θ' are positive numbers. Upon making the change of variable $t^2 = -\log y$, Laplace obtains the equation

$$(a.4.1) \qquad \int_{-\theta}^{\theta'} y(x)dx = \int_{-T}^{T'} e^{-t^2} \frac{dx}{dt}\, dt$$

where

$$(a.4.2) \qquad dx/dt = \sum_{j=0}^{\infty} (d^jU^{j+1}/dx^j)t^j/j! \quad .$$

Here T and T' are the values of $\left[-\log y(-\theta)\right]^{\frac{1}{2}}$ and $\left[-\log y(\theta')\right]^{\frac{1}{2}}$ respectively. Recall that under our assumptions, $\log f(x)$ has the expansion

$$(a.4.3) \qquad \log f(x) = -x^2/2 + a_3x^3 + a_4x^4 + \cdots \quad \text{for } |x| \leq \delta.$$

The coefficients $a_3, a_4, \cdots$ are given by (1.4.2). Using this expansion it is possible to establish (a.4.1) and (1.4.2) rigorously. We first make the change of variable $t_1^2 = -\log f(x)$ or more precisely $t_1 = x(\frac{1}{2} - a_3x - a_4x^2 - \cdots)^{\frac{1}{2}}$. Then since this transformation is analytic in some neighborhood of zero, the inverse function is given by the Lagrange series (see Markushevich (1965), Section 12)

$$\text{(a.4.4)} \qquad x = \sum_{j=1}^{\infty} \left[d^{j-1}(n^{\frac{1}{2}}U)^j/dx^{j-1}\right] \cdot t_1^j/j!$$

so that

$$\text{(a.4.5)} \qquad dx/dt_1 = \sum_{j=0}^{\infty} \left[d^j(n^{\frac{1}{2}}U)^{j+1}/dx^j\right] t_1^j/j!$$

in some neighborhood of $t_1=0$. Following this with the change of variable $t = n^{\frac{1}{2}}t_1$, we get (a.4.1) and (a.4.2).

Laplace (1847), Book One, Section 27, formula c states that

$$\text{(a.4.6)} \qquad \int_{-\Theta}^{\Theta'} y(x)dx = \left\{U + \frac{1}{2}\frac{d^2U^3}{2dx^2} + \cdots\right\} \int_{-T}^{T'} e^{-t^2}dt$$
$$+ \frac{1}{2} e^{-T^2}\left\{\frac{dU^2}{dx} - T\frac{d^2U^3}{2dx^2} + \cdots\right\}$$
$$- \frac{1}{2} e^{-T'^2}\left\{\frac{dU^2}{dx} + T'\frac{d^2U^3}{2dx^2} + \cdots\right\}$$

where Θ, Θ', T, and T' have the same relation as in (a.4.1). The right hand side may be obtained formally from (a.4.1) by first integrating termwise and then by parts. Consider first the case where $\Theta = \Theta' = \infty$. The expansion (a.4.6) gives

$$\text{(a.4.7)} \qquad \int_{-\infty}^{\infty} y(x)dx = \left\{U + \frac{1}{2}\frac{d^2U^3}{2dx^2} + \cdots\right\} \int_{-\infty}^{\infty} e^{-t^2}dt \quad .$$

In this case, the term by term integration may be shown to yield a valid asymptotic expansion by Watson's Lemma (see Copson (1965), page 49). As in Chapter 1, we would use Lemma 1.4.1 to neglect all but a small neighborhood of zero for the left hand side. Then this would, under the change of variable to t_1, transform to a neighborhood of zero, say $|t_1| \le 2\delta_1$, where the right hand side of (a.4.5) is a convergent series. That is

$$\int_{-\infty}^{\infty} y(x)dx = \int_{-\delta_1}^{\delta_1} e^{-nt_1^2} \frac{dx}{dt_1} dt_1 + O(n^{-M}) \quad (n \to \infty)$$

for every $M > 0$. Define a function $\phi(\cdot)$ by

$$\phi(t_1) = \begin{cases} dx/dt_1, \text{ for } |t_1| \le 2\delta_1 \\ 0 \quad , \text{ elsewhere} . \end{cases}$$

Then according to Watson's Lemma, $\int_0^{\infty} e^{-nt_1^2} \phi(t_1)dt$ has an asymptotic expansion, and the result (a.4.7) would follow. That is, the lemma shows that the terms obtained formally really are of the correct orders so that they do give a true asymptotic expansion. Copson (1965), page 3, credits Poincaré with originating the modern theory of asymptotic expansions in a paper written in 1886. Laplace then would not have had a precise definition of an expansion and would have just ordered the terms according to the power of n which enters them. Going back to the case where θ and θ' are not infinite, the situation is seen to be more complex, but it seems possible that the above technique would again show that the term by term integration would lead to an asymptotic expansion. However with T and T' as functions of n, it may be necessary to further expand the terms in (a.4.6). This is the case when we obtain (a.4.9) below.

Let us now evaluate U, dU^2/dx, and d^2U^3/dx^2 in terms of the derivatives of $\log f(x)$ given in (a.4.3). By definition,

$$v = xn^{-\frac{1}{2}} \left[-\log f(x)\right]^{-\frac{1}{2}}$$

so that we may also write

$$v = n^{-\frac{1}{2}}(\tfrac{1}{2} - a_3x - a_4x^2 - \cdots)^{-\frac{1}{2}}.$$

By straightforward calculation, we see that $U = 2^{\frac{1}{2}}n^{-\frac{1}{2}}$, $dU^2/dx = 2^2a_3n^{-1}$, and $d^2U^3/dx^2 = (3\cdot 2^{5/2}a_4 + 15\cdot 2^{3/2}a_3^2)n^{-3/2}$. According to (a.4.7), we have

$$\text{(a.4.8)} \qquad \int_{-\infty}^{\infty} y(x)dx = n^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\left[1 + (3a_4 + 15a_3^2/2)n^{-1} + \cdots\right] .$$

This agrees with our first correction term.

The other integral of interest is $\int_{-\infty}^{\xi n^{-\frac{1}{2}}} y(x)dx$. In this case, the upper limit of integration T' is equal to $\left[-n \log f(\xi n^{-\frac{1}{2}})\right]^{\frac{1}{2}}$. Upon expanding $\log f(\cdot)$, we have, taking $\xi > 0$ for convenience,

$$T' = \xi 2^{-\frac{1}{2}}\left[1 - a_3\xi n^{-\frac{1}{2}} + O(n^{-1})\right] \quad (n \to \infty)$$

where the O-function may depend on ξ . Writing

$$\int_{-\infty}^{T'} e^{-t^2}dt = \int_{-\infty}^{\xi 2^{-\frac{1}{2}}} e^{-t^2}dt + \int_{\xi 2^{-\frac{1}{2}}}^{\xi 2^{-\frac{1}{2}} - a_3 2^{-\frac{1}{2}}\xi^2 n^{-\frac{1}{2}}} e^{-t^2}dt + O(n^{-1}) \quad (n \to \infty)$$

and expanding the terms in the manner of Dorogocev (1962), we find that (a.4.6) gives

$$\text{(a.4.9)} \qquad \int_{-\infty}^{\xi n^{-\frac{1}{2}}} y(x)dx = n^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\left[\Phi(\xi) - \varphi(\xi)a_3(\xi^2+2)n^{-\frac{1}{2}}\right] + O(n^{-3/2}) \; (n \to \infty) .$$

Again the first correction term agrees with our result. With increasing difficulty, we could check the higher order terms.

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